FOURIER ANALYSIS METHODS FOR EVOLUTIONARY PARTIAL DIFFERENTIAL EQUATIONS. 17.03.2014

(I) Basic Fourier transform

$f \in L^1(\mathbb{R}^d)$, $\forall \xi \in \mathbb{R}^d$, $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx$

- $\hat{f} \in C_0(\mathbb{R}^d)$
- $f \in L^1$, $\hat{f} \in L^1$ $\Rightarrow$ Inversion Fourier formula:

$$f = \frac{1}{(2\pi)^d} \check{\mathcal{F}} f \quad \text{with} \quad \check{\mathcal{F}} f(\xi) = \int e^{ix\cdot\xi}\, \hat{f}(\xi)\,d\xi$$

Schwartz space:

$$\mathcal{S} = \{ f \in C^\infty(\mathbb{R}^d) : \forall k,\ \|f\|_{k,\mathcal{S}} < \infty \}$$

$$\|f\|_k = \max_{|\alpha|+|\beta| \le k} \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)|$$

For $f \in \mathcal{S}$:

- $\mathcal{F}f(\lambda\cdot)(\xi) = \frac{1}{\lambda^d} \mathcal{F}f\left(\frac{\xi}{\lambda}\right)$
- $L \in \mathrm{Aut}(\mathbb{R}^d)$: $\mathcal{F}(f \circ L) = \frac{1}{\det L} \mathcal{F}f \circ {}^T L^{-1}$
- $(i\xi)^\alpha \mathcal{F}f(\xi) = \mathcal{F}(\partial_x^\alpha f)(\xi)$
- $(i\partial_\xi^\alpha)\mathcal{F}f(\xi) = \mathcal{F}(x^\alpha f)(\xi)$
- $\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$

Thm: $\mathcal{F} : \mathcal{S} \to \mathcal{S}$

$$|\xi^\alpha \partial_\xi^\beta \mathcal{F}f(\xi)| \le |\mathcal{F}\partial_x^\alpha (x^\beta f)(\xi)| \le \|\partial_x^\alpha (x^\beta f)\|_{L^1} \le ~~C\|(1+|x|)^d~~$$

$$\le C \|(1+|x|)^d \partial_x^\alpha (x^\beta f)\|_{L^\infty}$$

Schwartz Idea: Extend $\mathcal{F}$ „by duality"

Dual set of $\mathcal{S}$: $\mathcal{S}'$ tempered distributions

Take $A : \mathcal{S} \to \mathcal{S}$

$T_A : \mathcal{S}' \to \mathcal{S}'$ defined for $f \in \mathcal{S}'$

$$\forall \varphi \in \mathcal{S},\ \langle {}^T A f, \varphi \rangle_{\mathcal{S}',\mathcal{S}} \overset{\text{def}}{=} \langle f, A\varphi \rangle_{\mathcal{S}',\mathcal{S}}$$

Example:

$A = \mathcal{F}$ $\quad \forall \varphi \in \mathcal{S}$, $\quad \langle {}^T\mathcal{F} f, \varphi \rangle \overset{def}{=} \langle f, \mathcal{F}\varphi \rangle$

If $f, \varphi \in \mathcal{S}$ then $\int f(\xi)\, \mathcal{F}\varphi(\xi)\, d\xi = \iint f(\xi)\varphi(x) e^{-ix\cdot\xi}\, dx\, d\xi$

$= \int \varphi(x)\, \mathcal{F}f(x)\, dx = \langle \mathcal{F}f, \varphi \rangle$

Conclusion: On $\mathcal{S}'$ $\mathcal{F}$ may be defined by

$\langle \mathcal{F}f, \varphi \rangle = \langle f, \mathcal{F}\varphi \rangle$

Consequence: For $f \in \mathcal{S}'$ we still have

$(i\xi)^\alpha \mathcal{F}f = \mathcal{F}(\partial_x^\alpha f)$ etc

Fourier - Plancherel theorem:

$\mathcal{F} : L^2 \to L^2$ (almost) - isometry

$\forall (f,g) \in L^2 \times L^2$, $\quad \int f(\xi)\bar{g}(\xi)\, d\xi = \frac{1}{(2\pi)^d} \int \mathcal{F}f(\xi)\, \overline{\mathcal{F}g(\xi)}\, d\xi$

Proof: $f, g \in \mathcal{S}$ (+ density on L^2)

$(\mathcal{F}f \mid \mathcal{F}g)_{L^2} = (f \mid \underbrace{\check{\mathcal{F}}(\mathcal{F}g)}_{(2\pi)^d \, Id}) = (2\pi)^d (f \mid g)$

~~$\check{\mathcal{F}}h(\xi) = \int$~~

(II) Sobolev spaces on $\mathbb{R}^d$

$\|f\|_{H^s} = \sqrt{\int \langle\xi\rangle^{2s} |\hat{f}(\xi)|^2 d\xi}$, $\quad \langle\xi\rangle = \sqrt{1+|\xi|^2}$

$H^s = \{ f \in \mathcal{S}' : \hat{f} \in L^2_{loc} \text{ and } \|f\|_{H^s} < \infty \}$

Special case: • $s = 0$, $H^0 = L^2$

• $s = 1$, $H^1 = \{ f \in L^2, \nabla f \in L^2 \}$

• $s \in \mathbb{N}$: $\|f\|_{H^s} \approx \sum_{|\alpha| \le s} \|\partial_\alpha f\|_{L_2}$

$\to H^s$ is a Hilbert space

$(f|g)_{H^s} = \int \langle\xi\rangle^{2s} \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, d\xi$

→ $\mathcal{S}$ and C_c^∞ dense in H^1

Proof: $(H^s)^\perp = \{0\}$

H^s may be localized: if $\varphi \in \mathcal{S}$ then $u \mapsto \varphi u$ maps H^s on H^s.

Duality: 1) H^s Hilbert space
Riesz representation thm. $(H^s)^* \simeq H^s$

2) Use distribution bracket or $\int fg\,dx$
Any $f \in (H^s)^*$ may be iadentified with some $g \in H^{-s}$ as follows:

$$\langle f, g \rangle_{(H^s)^*, H^s} = \langle f, g \rangle_{H^{-s}, H^s}$$

Interpolation inequality:

$f \in H^{s_0} \cap H^{s_1} \Rightarrow f \in H^s, \ \forall s \in [s_0, s_1]$

$$\|f\|_{H^s} \leq \|f\|_{H^{s_0}}^{\theta} \|f\|_{H^{s_1}}^{1-\theta} \quad \text{if} \quad s = \theta s_0 + (1-\theta) s_1$$

Proof: $|\hat{f}(\xi)|^2 \langle \xi \rangle^{2s} = \left(|\hat{f}(\xi)|^2 \langle \xi \rangle^{2s_0}\right)^{\theta} \left(|\hat{f}(\xi)|^2 \langle \xi \rangle^{2s_1}\right)^{1-\theta}$

Integrate + Hölder.

$f \longrightarrow f_\lambda, \quad x \mapsto f(\lambda x)$

Compare $\|f_\lambda\|_{H^s}$ and $\|f\|_{H^s}$

Not quite possible owing to $\sqrt{1 + |\xi|^2}$

Remove „1": Homogenous Sobolev semi-norms

$$\|u\|_{\dot{H}^s} = \sqrt{\int |\hat{u}(\xi)|^2 |\xi|^{2s} d\xi}$$

$$\|u_\lambda\|_{\dot{H}^s} = \lambda^{-d/2 + s} \|u\|_{\dot{H}^s}$$

Remark: $\|u\|_{\dot H^s}$ defined $\forall u \in \mathcal{S}'$ with $\hat u \in L^2_{loc}$

One has to be careful when defining $\dot H^s$

$\dot H^s = \{ u \in \mathcal{S}' : \hat u \in L^2_{loc}$ and $\|u\|_{\dot H^s} < \infty \}$.

With this definition, $\dot H^s$ is a Hilbert space if and only if $s < d/2$.

For $s < d/2$ this space coincides with $\overline{\mathcal{S}}^{\|\cdot\|_{\dot H^s}}$

Remark: $s \leq s' \Rightarrow H^{s'} \hookrightarrow H^s$

Not true that $\dot H^{s'} \hookrightarrow \dot H^s$

(iii) Sobolev embedding:

Goal: Compare $\|u\|_{L^p}$ and $\|u\|_{\dot H^s}$, $s > 0$

Can we have $\|u\|_{L^p} \leq C \|u\|_{\dot H^s}$, $\forall u \in \dot H^s$?

$u \longrightarrow u_\lambda \qquad u_\lambda(x) = u(\lambda x)$

$$\left.\begin{aligned} \|u_\lambda\|_{L^p} &= \lambda^{-d/p} \|u\|_{L^p} \\ \|u_\lambda\|_{\dot H^s} &= \lambda^{-d/p+s} \|u\|_{\dot H^s} \end{aligned}\right\} \quad \frac{d}{p} = \frac{d}{2} - s$$

Sobolev embedding: Let $s \in [0, \frac{d}{2})$ and $\frac{d}{p} = \frac{d}{2} - s$

then $\exists C > 0$, $\forall u \in \dot H^s$, $\|u\|_{L^p} \leq C \|u\|_{\dot H^s}$

(CRITICAL SOBOLEV EMBEDDING)

Remark: $\|u\|^p_{L^p} = p \int_0^{+\infty} |\{|u| \geq \lambda\}| \lambda^{p-1} d\lambda$

Fix $A > 0$: $\hat u = \underbrace{\mathbb{1}_{B(0,A)} \hat u}_{u_A^l} + \underbrace{\mathbb{1}_{{}^cB(0,A)} \hat u}_{u_A^h}$

$$\|u_A^l\|_\infty \leq \int_{B(0,A)} (|\hat u(\xi)| |\xi|^s) |\xi|^{-s} d\xi$$

$$\leq \|u\|_{\dot H^s} \sqrt{\int_{B(0,A)} |\xi|^{-2s} d\xi}$$

$$\|u_A^l\|_\infty \leq C A^{d/2 - s} \|u\|_{\dot H^s}$$

Assume WLOG, $\|u\|_{\dot H^s} = 1$

$$|\{|u| \geq \lambda\}| \leq |\{|u_A^l| \geq \lambda/2\}| + |\{|u_A^h| \geq \frac{\lambda}{2}\}|$$

Choose $A = A_\lambda$ s.t. $CA_\lambda^{d/2 - s} = \lambda/2$

$$\|u\|_{L^p}^p \leq p \int_0^\infty |\{|u_A^h| > \frac{\lambda}{2}\}| \lambda^{p-1} d\lambda$$

Markov inequality: $|\{|u_{A_\lambda}^h| > \frac{\lambda}{2}\}| \leq \frac{4 \|u_{A_h}^\lambda\|_{L^2}^2}{\lambda^2} = \frac{c}{\lambda^2} \|\mathbb{1}_{{}^cB(0,A)} \hat{u}\|_{L^2}^2$

$$\|u\|_{L^p}^p \leq C_p \int_0^\infty \int \lambda^{p-3} \mathbb{1}_{\{|\xi| \geq c\lambda^{p/d}\}} |\hat{u}(\xi)|^2 d\xi$$

$$\leq C \int_{\mathbb{R}^d} \left(\int_0^{c|\xi|^{d/p}} \lambda^{p-3} d\lambda \right) |\hat{u}(\xi)|^2 d\xi$$

$$\leq C \int_{\mathbb{R}^d} |\xi|^{d/p(p-2)} |\hat{u}(\xi)|^2 d\xi$$

$2s$

Corollary: • If $0 \leq s < \frac{d}{2}$, $H^s \hookrightarrow L^p$ for any $p \in [2, p_c]$

$\frac{d}{p_c} = \frac{d}{2} - s$

(use $H^s = L^2 \cap \dot H^s$ if $s \geq 0$)

• $s = \frac{d}{2}$: $H^s \hookrightarrow L^p$, $\forall p \in [2, \infty)$

• $s > \frac{d}{2}$ then $H^s \hookrightarrow \mathcal{F}L^1 (\hookrightarrow C_0)$

Dual Sobolev embedding:

• ~~$0 \leq s < \frac{d}{2}$~~ $q \in (1, 2]$.

Then $L^q \hookrightarrow \dot H^{-s}$ with $\frac{d}{q} = s + \frac{d}{2}$

Start of proof:

$$\|u\|_{\dot H^{-s}} = \sup_{\|v\|_{\dot H^s} = 1} \langle u, v \rangle$$

Sobolev emb: $\dot H^s \hookrightarrow L^p$ with $\frac{1}{p} + \frac{1}{q} = 1$

$|\langle u,v\rangle| \leq \|u\|_{L^q} \|v\|_{L^p} \leq C \|u\|_{L^q} \|v\|_{\dot{H}^s}$

<u>Corollary</u>: Gagliardo - Nirenberg inequality

$\forall u \in H^1(\mathbb{R}^d)$, $\|u\|_{L^p} \leq C \|u\|_{L^2}^{1-\theta} \|\nabla u\|_{L^2}^{\theta}$ with $\frac{d}{p} = \frac{d}{2} - \theta$

$\forall p \in [2, \frac{2d}{-(2-d)}] = [2, \frac{2d}{d-2}]$ and $p < \infty$

<u>Proof</u>: use $\dot{H}^s \hookrightarrow L^p$ with $-s + \frac{d}{2} = \frac{d}{p}$

$\|u\|_{L^p} \leq C \|u\|_{\dot{H}^s} \leq C \|u\|_{L^2}^{1-s} \|u\|_{\dot{H}^1}^{s} / \|\nabla u\|_{L^2}$