Compact Sobolev embedding:

Let $s > 0$, $p < p_c = \begin{cases} \frac{2d}{d-2s} & \text{if } s < d/2 \\ +\infty & s \geq d/2 \end{cases}$

Then $H^s \hookrightarrow L^p_{loc}$ compactly.

Proof: 1° $Id : H^s(\mathbb{R}^d) \to L^2(B(0,R))$ compact,

approximate Id by $T_\varepsilon : f \mapsto \chi_\varepsilon * f$

$\Rightarrow \| T_\varepsilon f - f \|_{L^2} \leq C \varepsilon^{2s} \| f \|_{H^s}$

hence $T_\varepsilon \to Id$ in $\mathcal{L}^2(H^s, L^2)$

$$\boxed{\begin{array}{l} \chi_\varepsilon = \varepsilon^{-d} \chi(\varepsilon^{-1} \cdot) \\ \hat{\chi} = 1 \text{ on } B(0,1) \\ 0 \leq \hat{\chi} \leq 1 \\ \operatorname{supp} \hat{\chi} \subset B(0,2) \end{array}}$$

$+ T_\varepsilon$ - Hilbert-Schmidt $\Rightarrow$ compact from $L^2(\mathbb{R}^d)$ to $L^2(B(0,R))$

2° Interpolate with critical Sobolev embedding

Dual statement: $1 \leq p \leq 2 \Rightarrow L^p \hookrightarrow H^{-s}_{loc}$ compactly if $s > s_c$ with $\frac{d}{p} = \frac{d}{2} + s_c$

Chapter 2: Incompressible Navier-Stokes equations

Introduction:

$$(NS_\nu) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

$u = u(t,x) \in \mathbb{R}^d$ - velocity field

$p = p(t,x) \in \mathbb{R}$ - pressure

Energy: $\int_{\mathbb{R}^d} u \cdot (NS_\nu)$

$$\int_{\mathbb{R}^d} u \cdot \partial_t u = \frac{1}{2} \frac{d}{dt} \| u \|^2_{L^2}$$

$$- \int \Delta u \cdot u = \| \nabla u \|^2_{L^2}$$

$$\int \nabla p \cdot u = - \int p \operatorname{div} u = 0$$

$$\int (u \cdot \nabla u) \cdot u = \sum_{i,j} \int u^j \partial_j u^i \cdot u^i$$

$$= -\frac{1}{2} \sum_{i,j} \int (u^i)^2 \partial_j u^j = 0$$

$$\frac{1}{2} \frac{d}{dt} \| u \|^2_{L^2} + \nu \| \nabla u \|^2_{L_2} = 0$$

Energy equality: $\| u(t) \|^2_{L^2} + 2\nu \int_0^t \| \nabla u \|^2_{L^2} dt = \| u(0) \|^2_{L^2}$

Expected: $u \in L^\infty([0,T]; L^2) \cap L^2([0,T]; H^1)$

(II) Leray theorem:

If u smooth then $(u \cdot \nabla u)^i = (\operatorname{div}(u \otimes u))^i = \sum_j \partial_j (u^i u^j)$

Def: of a weak solution:

$u \in L^2_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$

$\iint u \cdot \nabla \phi \, dx dt = 0, \quad \forall \phi \in C_c^\infty, \quad \forall \psi \in C_c^\infty, \operatorname{div} \psi = 0$

$$\int_{\mathbb{R}^d} u(t,x) \cdot \psi(t,x) \, dx + \int_0^t \int_{\mathbb{R}^d} (\nu \nabla u : \nabla \psi - (u \otimes u) : \nabla \psi - u \partial_t \psi)$$
$$= \int_{\mathbb{R}^d} u_0(x) \cdot \psi(0,x) dx$$

Leray thm: Let $u_0 \in L^2(\mathbb{R}^d)$ with $\operatorname{div} u_0 = 0$. Then (NS_ν) has a global weak solution s.t. $\forall t \in \mathbb{R}^+$,

$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u\|_{L^2}^2 \, d\tau \le \|u_0\|_{L^2}^2 .$$

Proof: Step 1: Construction of approximate ~~functio~~ solutions.

$$\widehat{J_n v}(\xi) = \mathbb{1}_{B(0,n)}(\xi) \hat{P_v}(\xi)$$

$P = Id + \nabla \operatorname{div}(-\Delta)^{-1}$

$\operatorname{div} P v = 0$

P - Helmholtz projection on div free vector fields

$$\hat{P_v}(\xi) = Id + \frac{\xi \xi}{|\xi|^2}$$

Consider $\frac{d}{dt} u = F_n(u) \overset{def}{=} \nu J_n \Delta u - J_n \operatorname{div}(J_n u \otimes J_n u) \quad (NS_{\nu,n})$

Claim: It is an ODE on L^2

J_n has range : $H^\infty = \bigcap_{s \in \mathbb{R}} H^s$

Cauchy - Lipschitz theorem implies $(NS_{\nu,n})$ has a unique maximal solution $u^n \in C^1([0,T_n); L^2)$

- $J_n^2 = J_n \Rightarrow J_n u^n$ is also a solution. Hence $u^n = J_n u^n$

Hence $u^n \in C^1([0,T_n), H^\infty)$ and $\operatorname{div} u^n = 0$.

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L^2}^2 = (\partial_t u^n \mid u^n)_{L^2}$$

$$-\nu\int_{\mathbb{R}^d} J_n \Delta u^n \cdot u^n dx = -\nu\int \Delta u^n \cdot u^n dx = \nu\|\nabla u^n\|_{L^2}^2$$

$$-\int J_n \operatorname{div}(u^n \otimes u^n)\cdot u^n = -\int \operatorname{div}(u^n\otimes u^n)\cdot u^n = 0$$

$$\forall t \in [0, T_n^*),\quad \|u^n(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla u^n\|_{L^2}^2 = \|J_n u_0\|_{L^2}^2 \leq \|u_0\|_{L^2}^2$$

Hence $T_n^* = +\infty$.

<u>Step 2:</u> Compactness

$$\frac{d}{dt}u^n = -\nu\Delta u^n - J_n \operatorname{div}(u^n\otimes u^n)$$

Claim: $\left(\frac{d}{dt}u^n\right)_{n\in\mathbb{N}}$ bounded in $L^p_{loc}(\mathbb{R}^+, H^{-1})$ for some $p>1$ if $d=2,3$.

Energy inequality $\Rightarrow u^n \in L^2(\mathbb{R}^+, \dot{H}^1) \Rightarrow \Delta u^n \in L^2(\mathbb{R}^+; \dot{H}^{-1})$

- <u>$d=2$</u>
$$\left.\begin{aligned} & u^n \in L^\infty(\mathbb{R}^+, L^2)\cap L^2(\mathbb{R}^+;\dot{H}^1) \\ & \|u\|_{L^4} \leq C\sqrt{\|u\|_{L^2}\|\nabla u\|_{L^2}}\end{aligned}\right\}\ u^n \text{ bounded in } L^4(\mathbb{R}^+\times\mathbb{R}^2)$$
$\Rightarrow \operatorname{div}(u^n\otimes u^n)$ in $L^2(\mathbb{R}^+; \dot{H}^{-1})$

- <u>$d=3$</u> $\|u_n\|_{L^4} \leq C\|u_n\|_{L_2}^{1/4}\|\nabla u_n\|_{L_2}^{3/4}$

$\Rightarrow u_n$ bounded in $L^{8/3}(\mathbb{R}^+; L^4)$

$\operatorname{div}(u^n\otimes u^n)$ in $L^{4/3}(\mathbb{R}^+; H^{-1})$

<u>Consequence:</u> $\exists\, \alpha>0$, u_n bounded in $C^\alpha_{loc}(\mathbb{R}^+; H^{-1})$

Consequence: $\exists\ \alpha > 0$

u_n bounded in $C^{\alpha}_{loc}(\mathbb{R}^+; H^{-1})$

u_n bounded in $L^{\infty}(\mathbb{R}^+; L^2)$

L^2 locally compact in H^{-1}. Apply Ascoli theorem on any $[0,T]$.

At the end, one gets $u \in L^{\infty}_{loc}(\mathbb{R}^+; H^{-1}_{loc})$ s.t.

$u_{\varphi(n)} \longrightarrow u$ in $L^{\infty}_{loc}(\mathbb{R}^+; H^{-1}_{loc})$

As u_n bounded in $L^{\infty}(\mathbb{R}^+, L^2) \cap L^2(\mathbb{R}^+, \dot{H}^1)$ one may assume that $u_{\varphi(n)} \rightharpoonup u$ in $L^{\infty}(\mathbb{R}^+, L_2) \cap L^2(\mathbb{R}^+, \dot{H}^1)$

It suffices to show that u is a weak solution of (NS_ν).

(II) Fujita - Kato theorem

$$\begin{cases} \partial_t u - \nu \Delta u = - u \cdot \nabla u - \nabla p \\ \operatorname{div} u = 0 \end{cases}$$

Getting rid of the pressure: Use $P = Id + \nabla \operatorname{div} (-\Delta)^{-1}$

$$\begin{cases} \partial_t u - \nu \Delta u = \underbrace{- P(\operatorname{div}(u \otimes u))}_{Q_{NS}(u,u)} \\ u|_{t=0} = u_0 \end{cases}$$

$$u(t) = e^{\nu t \Delta} u_0 + B(u,u)$$

with
$$\begin{cases} \partial_t B(u,u) - \nu \Delta B(u,u) = Q_{NS}(u,u) \\ u|_{t=0} = 0 \end{cases}$$

Method will work for (GNS_ν) : $\partial_t u - \nu \Delta u = Q(u,u)$

with $\widehat{(Q(v,w))}^i = \sum (\alpha^i_{j,k})(\xi)\, \widehat{v}^j \widehat{w}^k$

$\hookrightarrow$ homogeneous of degree 1.

<u>Abstract fixed point theorem:</u>

X Banach space, $B : X \times X \to X$ bilinear, continuous

$\forall\, v_0 \in X$, s.t. $4\|B\|\,\|v_0\|_X < 1 \Rightarrow \exists!\, v \in B_X(0, 2\|v_0\|_X)$,

$v = v_0 + B(v,v)$

<u>Proof:</u> Banach fixed point theorem.

$u(t) = \underbrace{e^{\nu t \Delta} u_0}_{v_0} + B(u,u)$. Find a good "$X$".

$X \subset \mathcal{S}'(\mathbb{R}^+ \times \mathbb{R}^d)$

<u>Scaling invariance of (GNS_ν)</u>

- u solution of (GNS_ν) with u_0

$\iff$

u_λ solution of (GNS_ν) with $u_{0,\lambda} : x \mapsto u_0(\lambda x)$

$u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$

Look for X with norm invariant by $u \mapsto u_\lambda$

<u>Examples:</u> $L^\infty(\mathbb{R}^+, \dot{H}^{\frac{d}{2}-1}) \cap L^2(\mathbb{R}^+, \dot{H}^{d/2})$ (energy space if $d=2$)

$L^4(\mathbb{R}^+, \dot{H}^{\frac{d-1}{2}})$

$L^\infty(\mathbb{R}^+; L^d)$

<u>Fujita - Kato theorem:</u> Let $u_0 \in \dot{H}^{d/2-1}(\mathbb{R}^d)$. Then (GNS_ν) has a unique maximal solution $u \in C([0,T^*), \dot{H}^{\frac{d}{2}-1}) \cap L^2_{loc}([0,T^*), \dot{H}^{d/2})$.

• If $\|u_0\|_{\dot H^{d/2-1}} \le c\nu$ then $T^* = +\infty$

and $\|u(t)\|_{\dot H^{d/2-1}}$ is nonincreasing ($\exists\, c>0$ - universal, dep only on dim.)

• If $T^* < +\infty$ then $\|u\|_{L^4([0,T^*),\, \dot H^{\frac{d-1}{2}})} = +\infty$

<u>Proof:</u> Solve $u = v_0 + B(u,u)$ with $v_0 = e^{\nu t \Delta} u_0$

$B(u,u)$ solution of $\begin{cases} \partial_t B(u,u) - \nu \Delta B(u,u) = Q(u,u) \\ B(u,u)|_{t=0} = 0 \end{cases}$

$X = L^4([0,T];\, \dot H^{\frac{d-1}{2}})$.

<u>Lemma:</u> $\begin{cases} \partial_t v - \nu \Delta v = f \\ v|_{t=0} = v_0 \end{cases}$ in $\mathbb{R}^+ \times \mathbb{R}^d$, $f \in L^2_{loc}(\mathbb{R}^+, \dot H^{s-1})$

$\exists!$ solution $v \in C(\mathbb{R}^+, \dot H^s) \cap L^2_{loc}(\mathbb{R}^+; \dot H^{s+1})$

$$\|v(t)\|^2_{\dot H^s} + 2\nu \int_0^t \|\nabla v\|^2_{\dot H^s} = \|v_0\|^2_{\dot H^s} + 2\int_0^t (f|v)_{\dot H^s}\, d\tau$$

$$\sqrt{\int_{\mathbb{R}^d} |\xi|^{2s} \sup_{t\in[0,T]} |\hat v(t,\xi)|^2\, d\xi} \le \|v_0\|_{\dot H^s} + \frac{1}{\sqrt{2\nu}} \sqrt{\int_0^T \|f\|^2_{\dot H^{s-1}}\, dt} \qquad (ii)$$

$$\nu^{1/p} \|v\|_{L^p_T(\dot H^{s+2/p})} \le \|v_0\|_{\dot H^s} + \frac{1}{\nu^{1/2}} \|f\|_{L^2(\dot H^{s-1})}$$

<u>Proof:</u> $\hat v(t,\xi) = e^{-\nu t|\xi|^2} \hat v_0(\xi) + \int_0^t e^{-\nu(t-\tau)|\xi|^2} \hat f(\tau,\xi)\, d\tau$

gives (ii)

<u>Return to the proof:</u> Apply lemma with $p=4$

$Q(u,u) \in L^2(\dot H^{d/2-2})$ $\quad (s = \frac{d}{2} - 1)$

$$\nu^{3/4} \|B(u,u)\|_{L^4} \le \|Q(u,u)\|_{L^2(\dot H^{d/2-2})} \le C \|u\|^2_{L^4(\dot H^{\frac{d-1}{2}})}$$

$\widehat{Q(u,w)}(\xi) \approx |\xi|\, \widehat{v\otimes w}$

Claim: $\|Q(v,w)\|_{\dot H^{d/2-2}} \le C\|v\|_{\dot H^{\frac{d-1}{2}}}\|w\|_{\dot H^{\frac{d-1}{2}}}$

Hence B maps $X \times X$ in X with $\|B\| \le \frac{C}{\nu^{3/4}}$

Abstract lemma $\Rightarrow$ If $4\|v_0\|_X \frac{C}{\nu^{3/4}} < 1$ then

$\exists! \, u \in B(0, 2\|v_0\|_X)$ satisfying $u = v_0 + B(u,u)$

Proof of claim: $d=2$ we have to prove

$$\|Q(v,w)\|_{\dot H^{-1}} \le C\|v\|_{\dot H^{1/2}}\|w\|_{\dot H^{1/2}}$$

$$\|Q(v,w)\|_{\dot H^{-1}} \le C\|v\otimes w\|_{L^2} \le C\|v\|_{L^4}\|w\|_{L^4} \le C\|v\|_{\dot H^{1/2}}\|w\|_{\dot H^{1/2}}$$

(Critical Sobolev embedding)

$d=3$ $\|Q(v,w)\|_{\dot H^{-1/2}} \le C\|v\otimes w\|_{\dot H^{1/2}}$

$$\begin{aligned}\|\operatorname{div}(v\otimes w)\|_{\dot H^{-1/2}} &\le C\|\operatorname{div}(v\otimes w)\|_{L^{3/2}}\\ &\le C(\|v\otimes\nabla w\|_{L^{3/2}} + \|w\otimes\nabla v\|_{L^{3/2}})\\ &\le C(\|v\|_{L^6}\|\nabla w\|_{L^2} + \|w\|_{L^6}\|\nabla v\|_{L^2})\\ &\le C\|\nabla v\|_{L^2}\|\nabla w\|_{L^2}\end{aligned}$$