$$(GNS_\nu) \quad \begin{cases} \partial_t u - \nu\Delta u + Q(u,u) = 0 \\ u|_{t=0} = u_0 \end{cases}$$

Fujita - Kato theorem: $u_0 \in \dot{H}^{d/2-1}$

(1) Then (GNS_ν) has a unique max. solution $u \in C([0,T^*), \dot{H}^{d/2-1}) \cap L^2_{loc}([0,T^*), \dot{H}^{d/2})$

(2) $\exists\, c>0$, $\|u_0\|_{\dot{H}^{d/2-1}} \le c\nu \Rightarrow T^* = +\infty$

In addition $t \mapsto \|u(t)\|_{\dot{H}^{d/2-1}}$ decays to 0

and $u \in L^2(\mathbb{R}^+, \dot{H}^{d/2})$

(3) Blow-up. $T^* < \infty \Rightarrow \int_0^{T^*} \|u(t)\|^4_{\dot{H}^{d/2-\frac{1}{2}}} dt = +\infty$.

Proof: Find $u \in X_T$ s.t. $u = v_0 + B(u,u)$, $X_T = L^4([0,T], \dot{H}^{\frac{d-1}{2}})$

and $v_0 = e^{\nu t\Delta} u_0$

we saw that $\exists!\, u \in X_T$ with $\|u\|_{X_T} \le 2\|v_0\|_{X_T}$

if $4C\|v_0\|_{X_T} \le \nu^{3/4}$ $(*)$, C- is the norm of B.

$$\|Q(v,w)\|_{\dot{H}^{\frac{d}{2}-2}} \le C\|v\|_{\dot{H}^{\frac{d-1}{2}}}\|w\|_{\dot{H}^{\frac{d-1}{2}}}$$

$$\left[\begin{array}{l} \partial_t u - \nu\Delta u = f, \quad u|_{t=0} = u_0 \\ \|u(t)\|^2_{\dot{H}^s} + 2\nu\int_0^t \|\nabla u\|^2_{\dot{H}^s} = \|u_0\|^2_{\dot{H}^s} + 2\int_0^t (f|u)_{\dot{H}^s} \\ \nu^{1/p}\|u\|_{L^p(0,T,\dot{H}^{s+2/p})} \le \|u_0\|_{\dot{H}^s} + \frac{1}{\nu^{1/2}}\|f\|_{L^2(0,T;\dot{H}^{s-1})} \end{array}\right]$$

$\nu^{1/4}\|v_0\|_{X_T^*} \le \|u_0\|_{\dot{H}^{\frac{d}{2}-1}}$. Hence $(*)$ fulfilled for all $T>0$ if $4C\|u_0\|_{\dot{H}^{\frac{d}{2}-1}} < \nu$ $(**)$

If $(**)$ is fulfilled then $\exists\, u \in X_\infty$ global solution to (E)

$$\partial_t u - \nu\Delta u = -Q(u,u)$$

$$\|Q(u,u)\|_{L_2(\mathbb{R}^+, \dot{H}^{\frac{d}{2}-2})} \le C\|u\|^2_{X_T^*}$$

Hence $\forall t \in \mathbb{R}^+$, $\|u(t)\|_{\dot H^{\frac d2 -1}} \leq \|u_0\|_{\dot H^{\frac d2 -1}} + \frac{C}{\nu^{1/2}} \|u\|^2_{X_\infty}$

$$\leq 2 \|u_0\|_{\dot H^{\frac d2 -1}}$$

Likewise, $\nu^{1/2} \|u\|_{L^2(\mathbb{R}^+, \dot H^{\frac d2})} \leq 2 \|u_0\|_{\dot H^{\frac d2 -1}}$

$$\|u(t)\|^2_{\dot H^{\frac d2 -1}} + 2\nu \int_0^t \|u\|^2_{\dot H^{\frac d2}} d\tau = \|u_0\|^2_{\dot H^{d/2-1}} + 2\int_0^t (Q(u,u)|u)_{\dot H^{\frac d2 -1}} d\tau$$

$$\leq \|u_0\|^2_{\dot H^{\frac d2 -1}} + 2\int_0^t \|Q(u,u)\|_{\dot H^{\frac d2 -2}} \|u\|_{\dot H^{\frac d2}} d\tau$$

$$\leq \quad - \| - \quad + 2C \int_0^t \underbrace{\|u\|_{\dot H^{\frac d2 -1}}}_{\leq 2\|u_0\|_{\dot H^{\frac d2 -1}}} \|u\|^2_{\dot H^{\frac d2}} d\tau$$

$\Rightarrow \|u(t)\|_{\dot H^{d/2-1}} \leq \|u_0\|_{\dot H^{d/2-1}}$

Resume to $4C \|e^{\nu t \Delta} u_0\|_{X_T} < \nu^{3/4}$ case u_0 large

$u_0 = u_0^l + u_0^h$

$\mathcal{F}(u_0^l) = \mathbb{1}_{B(0,\rho)} \mathcal{F} u_0$

We take ρ so that $8C \|u_0^h\|_{\dot H^{d/2-1}} < \nu$

Then $4C \|e^{\nu t \Delta} u_0\|_{X_T} < \frac{\nu^{3/4}}{2} + 4C \|e^{\nu t} u_0^l\|_{X_T}$

$$\|e^{\nu t \Delta} u_0^l\|_{X_T} = \|e^{\nu t \Delta} u_0^l\|_{L^4(0,T; \dot H^{\frac{d-1}{2}})} \leq T^{1/4} \|u_0^l\|_{\dot H^{\frac d2 - 1 + \frac12}}$$

$$\leq T^{1/4} \rho^{1/2} \|u_0\|_{\dot H^{\frac d2 -1}}$$

Take T s.t. $T^{1/4} \rho^{1/2} \|u_0\|_{\dot H^{\frac d2 -1}} = \frac{\nu^{3/4}}{2}$ then $4C \|v_0\|_{X_T} < \nu^{3/4}$

Stability estimate: Let u and v be 2 solutions of (GNS_ν) in $C([0,T]; \dot H^{\frac d2 -1}) \cap L^2([0,T], \dot H^{d/2})$. Let $w = v - u$. Then

$$\forall t \in [0,T], \quad \|w(t)\|^2_{\dot H^{\frac d2 -1}} + \nu \int_0^t \|w\|^2_{\dot H^{\frac d2}} \leq \|w_0\|^2_{\dot H^{\frac d2 -1}} \exp\left(\frac{C}{\nu^3} \int_0^t \|(u,v)\|^4_{\dot H^{\frac{d-1}{2}}}\right)$$

Proof: $\partial_t w - \nu \Delta w = Q(u,u) - Q(v,v) = Q(u+v, w)$

$$\|w(t)\|^2_{\dot H^{\frac d2 -1}} + 2\nu \int_0^t \|w\|^2_{\dot H^{d/2}} = \|w_0\|^2_{\dot H^{\frac d2 -1}} + 2\int_0^t (Q(u+v,w)|w)_{\dot H^{\frac d2 -1}}$$

$$\leq \|w_0\|^2_{\dot H^{d/2-1}} + 2\int_0^t \|Q(u+v,w)\|_{\dot H^{\frac d2 -2}} \|w\|_{\dot H^{\frac d2}}$$

$$\|Q(u+v,w)\|_{\dot H^{\frac d2-2}} \le C\|u+v\|_{\dot H^{\frac{d-1}{2}}}\|w\|_{\dot H^{\frac{d-1}{2}}}$$

$$\le C\|u+v\|_{\dot H^{\frac{d-1}{2}}}\sqrt{\|w\|^{1/2}_{\dot H^{\frac d2-1}}\|w\|^{1/2}_{\dot H^{d/2-1}}}\ \|w\|^{1/2}_{\dot H^{\frac d2}}$$

$$W(t) \le W(0) + 2C\int_0^t \nu^{-3/4}\|u+v\|_{\dot H^{\frac{d-1}{2}}}\|w\|^{1/2}_{\dot H^{\frac d2-1}}\left(\|w\|_{\dot H^{\frac d2}}\nu^{1/2}\right)^{3/2}$$

$$ab \le C_\varepsilon\frac{a^4}{4} + \frac34\varepsilon b^{4/3}$$

At the end $W(t) \le W(0) + \frac{C'}{\nu^3}\int_0^t \|u+v\|^4_{\dot H^{\frac{d-1}{2}}} W + ?\int_0^t \|w\|^2_{\dot H^{\frac d2}}$

\+ Gronwall

Remark: This gives uniqueness in Fujita-Kato thm

- If $d=2$ Leray solution is $L^\infty(\mathbb{R}^+, L^2)\cap L^2(\mathbb{R}^+, \dot H^1)$

 but $\partial_t u - \nu\Delta u = -P\operatorname{div}(u\otimes u) \in L^2(\mathbb{R}^+, \dot H^{-1})$

 Hence $u\in C(\mathbb{R}^+, L^2)$, hence it coincides with FK solutions.

Chapter 3: Littlewood - Paley decomposition

(I) Motivation, definition:

$$\mathcal{F}(\partial^\alpha u)(\xi) = (i\xi)^\alpha\,\hat u(\xi)$$

Bernstein inequalities: Assume $\operatorname{supp}\hat u \subset B(0, R\lambda)$

$\exists\, C = C_R$ s.t. $\forall k\in\mathbb{N}$, $\forall\ 1\le p\le q\le\infty$, $\|D^k u\|_{L^q} \le C^k\lambda^{k+\frac dp-\frac dq}\|u\|_{L^p}$

Reverse Bernstein: $\operatorname{Supp}\hat u \subset \{r\le|\xi|\le R\}$ then $\|D^k u\|_{L^p} \simeq \lambda^k\|u\|_{L^p}$

Proof: $v(x) = u(\lambda^{-1}x) \to$ reduces to $\lambda = 1$

(i) Let $\phi\in C_c^\infty(B(0,2R))$ with $\phi\equiv 1$ on $B(0,R)$

$\hat v = \phi\hat v \Rightarrow D^k v = D^k\mathcal{F}^{-1}\phi * v$

$\|D^k v\|_{L^q} \le \|D^k\mathcal{F}^{-1}\phi\|_{L^m}\|v\|_{L^p}$; $1+\frac1q = \frac1m+\frac1p$

(ii) Prove $\|v\|_{L^p} \le C\|Dv\|_{L^p}$

$\psi\in C_c^\infty(\{\frac r2\le|\xi|\le 2R\})$, $\hat v(\xi) = \psi\hat v(\xi) = \underbrace{-i\xi\frac{\psi(\xi)}{|\xi|^2}}_{\hat\theta(\xi)}\cdot\widehat{\nabla v}$

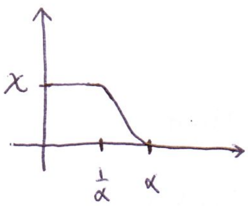
$$v = \Theta * \nabla v \quad \Rightarrow \quad \|v\|_{L^p} \le \|\Theta\|_{L^1} \|\nabla v\|_{L^p}$$

(II) Littlewood - Paley decomposition:

Aim: Splitting device for a general distribution $u \in \mathcal{S}'$ into pieces satisfying Bernstein ineq. assumptions.

Dyadic decomposition of unity

$\chi \,\cancel{\psi} \in C_c^\infty(B(0,\alpha))$ with $\cancel{\psi}\,\chi = 1$ on $B(0,\alpha^{-1})$, $\alpha > 1$



$0 \le \chi \le 1$, nonincreasing

$$\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$$

$$\forall \xi \in \mathbb{R}^d \setminus \{0\} \qquad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \qquad \text{(homogenous)}$$

$$\forall \xi \in \mathbb{R}^d, \quad \chi(\xi) + \sum_{j\in\mathbb{Z}_{\ge 0}} \varphi(2^{-j}\xi) = 1 \qquad \text{(nonhomogenous)}$$

Dyadic blocks:

$$\dot{\Delta}_j = \varphi(2^{-j}D), \quad j \in \mathbb{Z} \qquad \text{(homogenous)}$$

$$\begin{cases} \Delta_j = \dot{\Delta}_j & \text{if } j \ge 0 \\ \Delta_{-1} = \chi(D) \end{cases} \qquad \text{(nonhomogenous)}$$

$$\Delta_j = 0 \quad \text{if} \quad j \le -2$$

$$\mathcal{F}(A(D)u)(\xi) = A(\xi)\hat{u}(\xi)$$

$$A(D)u = \mathcal{F}^{-1}A * u$$

Ex: $\dot{\Delta}_j u = 2^{jd} h(2^j \cdot) * u$

$$h = \mathcal{F}^{-1}\varphi$$

Homogenous L^p decomposition of $u \in \mathcal{S}'$

$$\sum_{j\in\mathbb{Z}} \dot{\Delta}_j u$$

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \quad \text{in } \mathcal{S}'/\mathbb{R}[x_1,\dots,x_d] \qquad (*)$$

Remark: $(*)$ holds true in $\mathcal{S}'$ if $u \longrightarrow 0$ weakly at ∞.

Quasi-orthogonality: Take $\alpha = \frac{4}{3}$

$$\dot{\Delta}_j \dot{\Delta}_k = 0 \quad \text{if} \quad |j-k| > 1$$

Let $\dot{S}_j = \chi(2^{-j}D)$ (low frequency cutoff)

$$\dot{\Delta}_k(\dot{S}_{j-1} u \, \dot{\Delta}_j v) = 0 \quad \text{if} \quad |k-j| > 3$$

(III) Functional spaces: old and new

Sobolev spaces: $\|u\|^2_{\dot{H}^s} \simeq \sum_{j \in \mathbb{Z}} 2^{2js} \|\dot{\Delta}_j u\|^2_{L^2}$

$$\|u\|^2_{H^s} \simeq \sum_{j \geq -1} 2^{2js} \|\Delta_j u\|^2_{L^2}$$

Pf: $\|u\|^2_{\dot{H}^s} = \int |\xi|^{2s} |\hat{u}(\xi)|^2 \simeq \int \sum_j |\varphi(2^{-j}\xi)\hat{u}(\xi)|^2 |\xi|^{2s} d\xi$

$\sum_j \varphi(2^{-j}\xi) = 1$

$$\simeq \int \sum_j |\widehat{\dot{\Delta}_j u}(\xi)|^2 2^{2js} d\xi$$

$\frac{1}{2} \leq \sum_j \varphi^2(2^{-j}\xi) \leq 1$

Hölder spaces: $\|u\|_{\dot{C}^{0,s}} = \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^s}$, $s \in (0,1)$

$$\|u\|_{\dot{C}^{0,s}} \simeq \sup_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{L^\infty}, \quad \text{if} \quad u = \sum_j \dot{\Delta}_j u$$

Pf: $u \in \dot{C}^{0,s}$

$$\dot{\Delta}_j u(x) = 2^{jd} \int h(2^j(x-y)) u(y) dy =$$

$$= 2^{jd} \int h(2^j(x-y))(u(y) - u(x)) dy$$

$$|\dot{\Delta}_j u(x)| \leq 2^{jd} \int |h(2^j(x-y))| |x-y|^s dy \; \|u\|_{\dot{C}^{0,s}}$$

$$\leq 2^{-js} \| |x|^s h \|_{L^1} \|u\|_{\dot{C}^{0,s}}$$

Reverse inequality: Let $N^s(u) = \sup_j 2^{js} \|\dot{\Delta}_j u\|_{L^\infty} < \infty$

$$u(y) - u(x) = \sum_j (\dot{\Delta}_j u(y) - \dot{\Delta}_j u(x))$$

$$|u(y) - u(x)| \leq \sum_{j \leq j_0} |\dot{\Delta}_j u(y) - \dot{\Delta}_j u(x)| + \sum_{j > j_0}$$

$$\leq \sum_{j \leq j_0} |x-y| \|\nabla \dot{\Delta}_j u\|_{L^\infty} + 2 \sum_{j > j_0} \|\dot{\Delta}_j u\|_{L^\infty}$$

$$|u(x)-u(y)| \le C\,|x-y| \sum_{j\le j_0} 2^{j(1-s)} N^s(u) + 2N^s(u) \sum_{j>j_0} 2^{-js}$$

$$|u(y)-u(x)| \le C N^s(u)\left(|x-y|\, 2^{j_0(1-s)} + 2^{-j_0 s}\right)$$ Choose j_0 s.t. $|x-y|2^{j_0} \approx 1$

Homogeneous Besov seminorm:

$$\|u\|_{\dot B^s_{p,q}} = \left\| 2^{js} \|\dot\Delta_j u\|_{L^p(\mathbb{R}^d)} \right\|_{\ell^q(\mathbb{Z})}$$

$$\dot B^s_{p,q} = \left\{ u \in \mathcal{S}' : \|u\|_{\dot B^s_{p,q}} < \infty \text{ and } \|\dot S_j u\|_{L^\infty} \xrightarrow{j\to 0} 0 \right\}$$

Nonhomogeneous Besov space:

$$B^s_{p,q} = \left\{ u \in \mathcal{S}' : \|u\|_{B^s_{p,q}} < \infty \right\}, \quad \|u\|_{B^s_{p,q}} = \left\| 2^{js} \|\Delta_j u\|_{L^p(\mathbb{R}^d)} \right\|_{\ell^q(\mathbb{Z})}$$

Ex: $H^s = B^s_{2,2}$ and $C^{0,s} = B^s_{\infty,\infty}$

Lemma: Let $(u_j)_{j\ge -1}$ with (i) $\operatorname{supp} \hat u_j \subset 2^j\, \mathcal{C}(0,r,R)$ — $\mathcal{C}$-notation for annulus

and $N(u) \overset{\text{def}}{=} \left\| 2^{js} \|u_j\|_{L^p(\mathbb{R}^d)} \right\|_{\ell^q(\mathbb{N}\cup\{-1\})} < \infty$

Then $u = \sum_{j\ge -1} u_j \in B^s_{p,q}$ and $\|u\|_{B^s_{p,q}} \le C_{r,R,s}\, N(u)$

(ii) If $\operatorname{supp} \hat u_j \subset 2^j B(0,R)$ and $s>0$ then it is still true

Proof: $\Delta_k u = \sum_{|j-k|\le K} \Delta_k u_j$; K- dependent only on the annulus

$\|\Delta_k u_j\|_{L^p} \le C \|u_j\|_{L^p}$

$$2^{ks} \|\Delta_k u\|_{L^p} \le C \sum_{|j-k|\le K} 2^{(k-j)s} \cdot 2^{js} \|u_j\|_{L^p} \le (2K+1) C\, 2^{K|s|}\, 2^{js} \|u_j\|_{L^p}$$

Then take ℓ^q norm

(ii) $$\Delta_k u = \sum_{k\le j+K} \Delta_k u_j$$

$$2^{ks} \|\Delta_k u\|_{L^p} \le C \sum_{j\ge k-K} 2^{(k-j)s}\, 2^{js} \|u_j\|_{L^p}$$ Take ℓ^q norm and use $\ell^1 * \ell^q \to \ell^q$ OK, $s>0$

(IV) Embeddings: (1) $\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty}$

(2) $\dot{B}^s_{p,r_1} \hookrightarrow \dot{B}^{s+\frac{d}{q}-\frac{d}{p}}_{q,r_2}$ if $1 \le r_1 \le r_2 \le \infty$, $1 \le p \le q \le \infty$

•) $B^0_{p,1} \hookrightarrow L^p \hookrightarrow B^0_{p,\infty}$

$B^s_{p,r_1} \hookrightarrow B^{s+\frac{d}{q}-\frac{d}{p}-a}_{q,r_2}$ $1 \le p \le q \le \infty$, $a \ge 0$, $1 \le r_1 \le r_2 \le \infty$

or $a > 0$ and r_1, r_2 arbitrary

Scaling invariance:

$u_\lambda(x) = u(\lambda x)$

$\|u_\lambda\|_{\dot{B}^s_{p,q}} \approx \lambda^{s-\frac{d}{p}} \|u\|_{\dot{B}^s_{p,q}}$

(1) and (2) are "critical embeddings". Both sides have the same scaling invariance.

Proof: Let $u \in L^p$, then $\|\dot{\Delta}_j u\|_{L^p} \le \|h\|_{L^1} \|u\|_{L^p}$ $\forall j \in \mathbb{Z}$

Let $u \in \dot{B}^0_{p,1}$, then $u = \sum \dot{\Delta}_j u$, thus $\|u\|_{L^p} \le \underbrace{\sum \|\dot{\Delta}_j u\|_{L^p}}_{\dot{B}^0_{p,1}}$

Proof of emb. (2):

$u \in \dot{B}^s_{p,r_1} \hookrightarrow \dot{B}^s_{p,r_2}$

$\|\dot{\Delta}_j u\|_{L^q} \le \cancel{2^j}.\ 2^{j(\frac{d}{p}-\frac{d}{q})} \|\dot{\Delta}_j u\|_{L^p}$ (Bernstein)

Multiply by $2^{j(s+\frac{d}{q}-\frac{d}{p})}$ + ℓ^{r_2} summation

(V) Nonlinear estimates:

$u \in \mathcal{S}'$, $v \in \mathcal{S}'$

When are we allowed to do the product?

Bony's idea: $u \cdot v = \sum_{j,k} \Delta_j u \Delta_k v$

$$uv = \sum_{k \le j-N} \Delta_j u \Delta_k v + \sum_{j \le k-N} \Delta_j u \Delta_k v + \sum_{|j-k|<N} \Delta_j u \Delta_k v$$

<u>Remark:</u> $\displaystyle\sum_{k\le j-N} \Delta_k = S_{j-N+1} = \chi(2^{-(j-N+1)} D)$

$$uv = \underbrace{\sum_j S_{j-N+1} v \, \Delta_j u}_{T_v u} + \underbrace{\sum_j S_{j-N+1} u \, \Delta_j v}_{T_u v} + \underbrace{\sum_{|j-k|<N} \Delta_j u \, \Delta_k v}_{R(u,v)}$$

paraproducts

Remainder

If $\alpha = \frac{4}{3}$ one may take $N=2$

<u>Continuity of T and R:</u>

$T : L^\infty \times B^s_{p,r} \longrightarrow B^s_{p,r} \qquad \forall s, \forall p, \forall r$

$T : B^{-t}_{\infty,\infty} \times B^s_{p,r} \longrightarrow B^{s-t}_{p,r} \qquad \forall\, t>0$

$R : B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2} \longrightarrow B^{s_1+s_2}_{p,r} \qquad$ if $s_1+s_2>0$

$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{r} \le \frac{1}{r_1} + \frac{1}{r_2}$

$R : B^{s}_{p_1,r_1} \times B^{-s}_{p_2,r_2} \longrightarrow B^{0}_{p,\infty} \qquad$ if $\frac{1}{r_1} + \frac{1}{r_2} \ge 1, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$

<u>Proof:</u> $T_u v = \displaystyle\sum_j S_{j-1} u \, \Delta_j v \qquad \operatorname{Supp} \mathcal{F}(S_{j-1} u \, \Delta_j v) \subset 2^j C(0, r, R)$

$\| S_{j-1} u \, \Delta_j v \|_{L^p} \le \| S_{j-1} u \|_{L^\infty} \, \| \Delta_j v \|_{L^p}$

$S_{j-1} u = 2^{(j-1)d} (\mathcal{F}^{-1} \chi)(2^j \cdot) * u$

$\| S_{j-1} u \|_{L^\infty} \le C \| u \|_{L^\infty}$

Therefore $\quad 2^{js} \| S_{j-1} u \, \Delta_j v \|_{L^p} \le C \|u\|_{L^\infty} \left(2^{js} \| \Delta_j v \|_{L^p} \right)$

If $\quad u \in B^{-t}_{\infty,\infty}$

$S_{j-1} u = \displaystyle\sum_{k\le j-2} \Delta_k u$

$\| S_{j-1} u \|_{L^\infty} \le \displaystyle\sum_{k\le j-2} 2^{kt} \| \Delta_k u \|_{L^\infty} 2^{-kt} \le C\, 2^{jt} \| u \|^{-t}_{\infty,\infty}$

$$R(u,v) = \sum_j \Delta_j u \, \Delta_j v$$

$$\operatorname{Supp}(\Delta_j u \, \Delta_j v) \subset 2^j B(0,R)$$

$$2^{j(s_1+s_2)} \|\Delta_j u \, \Delta_j v\|_{L^p} \leq \left(2^{js_1} \|\Delta_j u\|_{L^{p_1}}\right)\left(2^{js_2} \|\Delta_j v\|_{L^{p_2}}\right)$$

$$\| \quad - \quad \| \quad - \quad \|_{\ell^r} \leq \| 2^{js_1} \|\Delta_j u\|_{L^{p_1}} \|_{\ell^{r_1}} \| 2^{js_2} \|\Delta_j v\|_{L^{p_2}} \|_{\ell^{r_2}} \quad \text{if } \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$$

As $s_1 + s_2 > 0$ one may apply the lemma

<u>Tame estimates:</u>

$$\|uv\|_{B^s_{p,r}} \lesssim \|u\|_{L^\infty} \|v\|_{B^s_{p,r}} + \|v\|_{L^\infty} \|u\|_{B^s_{p,r}} \qquad \forall s>0, \ \forall p,r$$

$$uv = \underset{\substack{\uparrow \\ L^\infty}}{T_u} \underset{\substack{\uparrow \\ B^s_{p,r}}}{v} + \underset{\substack{\uparrow \\ L^\infty}}{T_v} \underset{\substack{\uparrow \\ B^s_{p,r}}}{u} + \underset{?}{R(u,v)}$$

Do we have $\|R(u,v)\|_{B^s_{p,r}} \leq C \|u\|_{L^\infty} \|v\|_{B^s_{p,r}}$?

We know $R : B^0_{\infty,\infty} \times B^s_{p,r} \longrightarrow B^s_{p,r}$ if $s>0$

and $L^\infty \hookrightarrow B^0_{\infty,\infty}$

<u>Sobolev estimate:</u> $\|uv\|_{H^{s+t-\frac{d}{2}}} \lesssim \|u\|_{H^s} \|v\|_{H^t}$

if $s+t>0$, $s<\frac{d}{2}$, $t<\frac{d}{2}$

Also work with $\dot{H}$

<u>Proof:</u> $uv = T_u v + T_v u + R(u,v)$

$H^s \hookrightarrow B^{s-\frac{d}{2}}_{\infty,\infty}$ and $s - \frac{d}{2} < 0$

Hence $T : H^s \times H^t \longrightarrow H^{s+t-\frac{d}{2}}$

Similarily $H^t \hookrightarrow B^{t-\frac{d}{2}}_{\infty,\infty}$ and $t-\frac{d}{2} < 0$

$$R : H^s \times H^t = B^s_{2,2} \times B^t_{2,2} \longrightarrow B^{s+t}_{1,1} \hookrightarrow B^{s+t-d/2}_{2,1} \hookrightarrow B^{s+t-\frac{d}{2}}_{2,2}$$

$s+t>0$