(E) $\begin{cases} \partial_t u + u\cdot\nabla u + \nabla\pi = 0 \\ \operatorname{div} u = 0 \end{cases}$ in $\mathbb{R}^2$

(B) $\begin{cases} \partial_t u + u\cdot\nabla u + \nabla\pi = \theta e_2 \\ \partial_t \theta + u\cdot\nabla\theta = 0 \\ \operatorname{div} u = 0 \end{cases}$

(IE) $\begin{cases} \rho(\partial_t u + u\cdot\nabla u) + \nabla\pi = 0 \\ \partial_t \rho + u\cdot\nabla\rho = 0 \\ \operatorname{div} u = 0 \end{cases}$

Valicity: $\omega = \partial_1 u^2 - \partial_2 u^1$

Transport equation $\partial_t \omega + u\cdot\nabla\omega = 0$

$\|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}$

Wolibner - Yudovich: $\omega_0 \in L^\infty \cap L^1 \Rightarrow \exists!$ global sol. with $\|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}$

(B) $\to$ $\partial_t \omega + u\cdot\nabla\omega = \partial_1\theta$

(IE) $\to$ ugly

(II) Transport equation

(T) $\begin{cases} \partial_t a + v\cdot\nabla a = f \\ \operatorname{div} v = 0 \end{cases} \Rightarrow \|a(t)\|_{L^p} \le \|a_0\|_{L^p} + \int_0^t \|f\|_{L^p}\, d\tau$

$a_0 \in \dot{C}^{0,\varepsilon}$ $\qquad \psi$: flow of v, $\varepsilon \in (0,1]$

$f = 0$, $\quad a(t,x) = a_0(\psi_t^{-1}(x))$

$$|a(t,y) - a(t,x)| = |a_0(\psi_t^{-1}(y)) - a_0(\psi_t^{-1}(x))|$$
$$\le \|a_0\|_{\dot{C}^{0,\varepsilon}} \|\nabla\psi_t^{-1}\|_{L^\infty}^{\varepsilon} |y-x|^\varepsilon$$
$$\|\nabla\psi_t^{-1}\|_{L^\infty} \le \exp\left(\int_0^t \|\nabla v\|_{L^\infty} d\tau\right)$$

„General estimate for (T)"

X - reasonable space

$$\|a(t)\|_X \leq \|a_0\|_X \exp\left(\int_0^t \|\nabla v\|_{L^\infty} d\tau\right)$$

$$B^0_{\infty,1} \hookrightarrow C_b$$

Littlewood - Paley decomposition $(\Delta_j)_{j \geq -1}$

$$u = \sum_{j \geq -1} \Delta_j u \quad \text{in } \mathcal{S}'$$

Δ_j - Fourier truncation op. about frequency 2^j

$$\|u\|_{B^0_{\infty,1}} = \sum_j \|\Delta_j u\|_{L^\infty}$$

$$\partial_t a + v \cdot \nabla a = 0$$
$$a|_{t=0} = a_0 \in B^0_{\infty,1}$$
$$\operatorname{div} v = 0$$

Hmidi - Keraani dynamic interpolation

$$a = \sum_j a_j \quad \text{with} \quad \begin{cases} \partial_t a_j + v \cdot \nabla a_j = 0 \\ a_j|_{t=0} = \Delta_j a_0 \end{cases}$$

$$\|a(t)\|_{B^0_{\infty,1}} \leq \sum_j \|a_j(t)\|_{B^0_{\infty,1}} \leq \sum_{j,k} \|\Delta_k a_j(t)\|_{L^\infty}.$$

Fix $N \in \mathbb{N}$

$$\|a(t)\|_{B^0_{\infty,1}} \leq \sum_{|j-k| \leq N} \|\Delta_k a_j(t)\|_{L^\infty} + \sum_{|j-k|>N}$$

$$\lesssim N \sum_j \|a_j(t)\|_{L^\infty} + \sum_{|j-k| \gtrless N} 2^{-\frac{k}{2}} \|\cancel{\Delta_k} a_j(t)\|_{C^{0,\frac{1}{2}}}$$

$$\|f\|_{C^{0,\varepsilon}} \approx \sup_j 2^{j\varepsilon} \|\Delta_j u\|_{L^\infty}$$

$$\|a_j(t)\|_{L^\infty} = \|\Delta_j a_0\|_{L^\infty}$$

$$\|a_j(t)\|_{C^{0,\frac{1}{2}}} \leq \|\Delta_j a_0\|_{C^{0,\frac{1}{2}}} \left(\exp \int_0^t \|\nabla v\|_{L^\infty}\right) \approx 2^{j/2} \|\Delta_j a_0\|_{L^\infty} \exp(\ldots)$$

$$\|a(t)\|_{B^0_{\infty,1}} \lesssim N \sum_j \|\Delta_j a_0\|_{L^\infty} + \left(\sum_{|j-k|>N} 2^{-\frac{|j-k|}{2}} \|\Delta_j a_0\|_{L^\infty}\right) \exp \int_0^t \|\nabla v\|_{L^\infty} d\tau$$

Estimate in $C^{0,-\frac{1}{2}}$?

$$\|a(t)\|_{B^0_{\infty,1}} \lesssim \|a_0\|_{B^0_{\infty,1}} \left(N + 2^{-N/2} \left(\exp \int_0^t \|\nabla v\|_{L^\infty}\right)\right)$$

Take N s.t. $2^{N/2} \simeq \exp \int_0^t \|\nabla v\|_{L^\infty}$

$$\Rightarrow \|a(t)\|_{B^0_{\infty,1}} \lesssim \underbrace{\|a_0\|_{B^0_{\infty,1}}}_{+\int_0^t \|f\|_{B^0_{\infty,1}}} \left(1 + \int_0^t \|\nabla v\|_{L^\infty}\right) \quad *$$

Thm: (E) is globally well-posed for $\omega_0 \in B^0_{\infty,1} \cap L^1$

Proof: $\|\omega(t)\|_{B^0_{\infty,1}} \leq \|\omega_0\|_{B^0_{\infty,1}} \left(1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau\right)$

Biot - Savait $\quad u = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} * \omega = \nabla^\perp (-\Delta)^{-1} \omega$

$$\mathcal{F}(\nabla^\perp (-\Delta)^{-1} \omega)(\xi) = \frac{\xi^\perp}{|\xi|^2} \hat{\omega}(\xi)$$

$$\nabla u = \underbrace{\Delta_{-1}}_{\text{LF cut-off}} \nabla u + \underbrace{\nabla \nabla^\perp (-\Delta)^{-1} (Id - \Delta_{-1}) \omega}_{d^\circ 0 \text{ away from } 0}$$

LF- low frequency $\qquad B^0_{\infty,1} \longrightarrow B^0_{\infty,1}$ $\qquad \|\nabla u\|_{L^\infty} \lesssim \|\omega\|_{L^1 \cap B^0_{\infty,1}}$

$$\|u\|_{L^\infty} \leq C \sqrt{\|\omega\|_{L^1} \|\omega\|_{L^\infty}}$$

$$\Delta_{-1} \nabla u = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} * \Delta_{-1} \nabla \omega \Rightarrow \|\Delta_{-1} \nabla u\|_{L^\infty} \lesssim \sqrt{\|\Delta_{-1} \nabla \omega\|_{L^1} \|\Delta_{-1} \nabla \omega\|_{L^\infty}}$$

$$\lesssim \|\Delta_{-1} \omega\|_{L^1} \lesssim \|\omega\|_{L^1}$$

$$\|\omega(t)\|_{B^0_{\infty,1} \cap L^1} \leq C \|\omega_0\|_{B^0_{\infty,1} \cap L^1} \left(1 + \int_0^t \|\omega\|_{B^0_{\infty,1} \cap L^1}\right)$$

Gronwall:

$$\|\omega(t)\|_{B^0_{\infty,1} \cap L^1} \leq C \|\omega_0\|_{B^0_{\infty,1} \cap L^1} \exp\left(t \|\omega_0\|_{B^0_{\infty,1} \cap L^1}\right)$$

$$\tilde{B}^0 = B^0_{\infty,1} \cap L^1$$

Boussinesq $\begin{cases} \partial_t \theta + u\cdot\nabla\theta = 0 \\ \partial_t \omega + u\cdot\nabla\omega = \partial_1\theta \end{cases}$ $\quad \operatorname{div} u = 0$

$$\|\omega(t)\|_{\tilde{B}^0} \leq \left(\|\omega_0\|_{\tilde{B}^0} + \int_0^t \|\partial_1\theta\|_{\tilde{B}^0}\right)\left(1 + \int_0^t \|\omega\|_{\tilde{B}^0}\right)$$

$$\partial_t \partial_j \theta + u\cdot\nabla\partial_j\theta = -\partial_j u\cdot\nabla\theta$$

$$\|\partial_j\theta(t)\|_{\tilde{B}^0} \leq \left(\|\partial_j\theta_0\|_{\tilde{B}^0} + \int_0^t \|\partial_j u\cdot\nabla\theta\|_{\tilde{B}^0}\right)\left(1 + \int_0^t \|\omega\|\right)$$

Claim: $\quad \|\partial_j u\cdot\nabla\theta\|_{B^0_{\infty,1}} \lesssim \|\partial_j u\|_{B^0_{\infty,1}} \|\nabla\theta\|_{B^0_{\infty,1}}$

Bony's decomposition:

$$\partial_j u\cdot\nabla\theta = \underbrace{T_{\partial_j u}}_{L^\infty} \underbrace{\nabla\theta}_{B^0_{\infty,1}} + \underbrace{T_{\nabla\theta}}_{L^\infty} \underbrace{\partial_j u}_{B^0_{\infty,1}} + R(\partial_j u, \nabla\theta)$$

$$R(\partial_j u,\nabla\theta) = \operatorname{div} R(\underbrace{\partial_j u}_{L^\infty}, \underbrace{\theta}_{B^1_{\infty,1}})$$

Conclusion: $\quad \|\theta(t)\|_{B^1_{\infty,1}} \leq \|\theta_0\|_{B^1_{\infty,1}} \exp\left(c\int_0^t \|\omega\|_{\tilde{B}^0}\right)$

Notation: $\quad \Omega(t) = \|\omega(t)\|_{\tilde{B}^0}$

$\quad \Theta(t) = \|\theta(t)\|_{B^1_{\infty,1}}$

$$\Omega(t) \leq \left(\Omega_0 + \Theta_0\, t\, e^{-\int_0^t \Omega}\right)\left(1 + \int_0^t \Omega\right)$$

Assume: $\quad \Theta_0 T e^{\int_0^T \Omega} \leq \Omega_0 \quad$ then $\quad \Omega(t) \leq 2\Omega_0 e^{2\Omega_0 t}$ on $[0,T]$

$(\mathcal{H})$

$(\mathcal{H})$ satisfied if $\quad \Theta_0 T e^{(e^{2\Omega_0 T}-1)} \leq \Omega_0$

that is $\quad 2\Omega_0 T(e^{e^{2\Omega_0 T}} - 1) \leq \dfrac{2\Omega_0^2}{\Theta_0}$

it suffices $\quad e^{2e^{2\Omega_0 T}} - 1 \leq \dfrac{2\Omega_0^2}{\Theta_0}$

$$T \leq \underbrace{\frac{1}{2\Omega_0} \log\left(1 + \frac{1}{2}\log\left(1 + \frac{2\Omega_0^2}{\Theta_0}\right)\right)}_{T_0}$$

If $\|\Theta_0\|_{B^1_{\infty,1}} \to 0$ then $T \to +\infty$

(IE) $\partial_t \rho + u\cdot\nabla \rho = 0$

$$\partial_t \omega + u\cdot\nabla\omega + \nabla\left(\frac{1}{\rho}\right) \wedge \underbrace{\nabla\pi}_{u^2} = 0$$

$$\Delta\pi = \dots$$