Mamy $\forall p \geq 1 \quad \underset{\|Q^TR - Id\|^2_{L_p(S)}}{\overset{\vee\!\!/}{C\|R-Q\|^2_{L_p(S)}}} \leq C\left(\|R - \oint R\|^2_{L_p(S)} + \underbrace{\|\oint R - Q\|^2}\right) \leq \frac{c}{h^3}\int_{S^h} \mathrm{dist}^2(\nabla u, SO(3))$

$$\leq C\int_S \mathrm{dist}^2(\oint R, SO(3))$$

$$\leq C\int_S |\oint R - R(x)|^2 dx$$

■

20.12.2012

Lemat: $\forall h \ \exists Q^h \in SO(3),\ c^h \in \mathbb{R}^3,\ (u^h \in W^{1,2}(S^h, \mathbb{R}^3)) \quad \|u^h - (Q^h x + c^h)\|_{W^{1,2}(S^h)} \leq C h^{-1/2} E_h(u^h)^{1/2}$

Dowód: $E^h(u^h) \geq \frac{c}{h}\int_{S^h} \mathrm{dist}^2(\nabla u^h, SO(3)) \underset{\text{tw. aproks.}}{\geq} \frac{c}{h}\int_{S^h} |\nabla u^h - R^h_\pi|^2$

$R^h \in W^{1,2}(S, SO(3))$

$Q^h \in SO(3)$

$$\geq \frac{c}{h}\left(\frac{1}{2}\int_{S^h} |\nabla u^h - Q^h|^2 - \int_{S^h} |R^h_\pi - Q^h|^2\right)$$

$$\geq \frac{c}{h}\int_{S^h} |\nabla u^h - Q^h|^2 - \underbrace{\frac{c}{h^2}\frac{1}{h}\int_{S^h} \mathrm{dist}^2(\nabla u^h, SO(3))}_{\leq \frac{c}{h^2} E^h(u^h)}$$

Uwaga: Możemy zamiast R^h wziąć $\tilde{R}^h$ i wziąć Q^h t.że

~~dist~~ $|Q^h - \oint \tilde{R}^h| = \mathrm{dist}(\oint \tilde{R}^h, SO(3))$

[nigdzie nie używamy małości energii]

więc

$$\frac{1}{h}\int_{S^h} |\nabla u^h - Q^h|^2 \leq \frac{c}{h^2} E^h(u^h). \qquad \text{(z nier. Poincare)}$$

$$\frac{1}{h}\|u^h - (Q^h x + c^h)\|^2_{W^{1,2}} \leq \frac{c}{h^2} E^h(u^h)$$

■

Mamy: $\frac{1}{h}\int_{S^h} f^h(u^h - id)\,dx = h^\alpha\left(\frac{1}{h}\int_{S^h} \det(Id + t\Pi)^{-1}\langle f(x), u^h - x\rangle\,dz\right) \underbrace{\pm Q^h x + c^h}$

$$= h^\alpha\Big(\frac{1}{h}\int_{S^h} \det(Id + t\Pi)^{-1}\langle f(x), u^h - (Q^h z + c^h)\rangle\,dz +$$

$$+ \underbrace{\frac{1}{h}\int_{S^h}\int_{-h/2}^{h/2} \langle f(x), Q^h x + c^h - x\rangle\,dz}_{\substack{\text{całk. się do } x \\ \leq 0}}$$

$$\leq C h^{\alpha-1} \|f\|_{L^2(S^h)} \|u^h - (Q^h x + c^h)\|_{L^2(S^h)}$$

$$\overset{\text{lemat}}{\leq} C h^{\alpha-1} h^{1/2} h^{-1/2} E^h(u^h)^{1/2}.$$

<u>Stąd:</u> $J^h(u^h) \geq E^h(u^h) - C h^{\alpha-1} E^h(u^h)^{1/2} \quad /\!/ \cdot h^{\frac{1}{2\alpha-2}}$

$$\frac{1}{h^{2\alpha-2}} J^h(u^h) \geq \frac{1}{h^{2\alpha-2}} \overbrace{}^{E^h} - C\left(\frac{1}{h^{2\alpha-2}} E^h(u^h)\right)^{1/2}$$

czyli $\left(\frac{1}{h^\beta} J^h\right) \geq \frac{1}{h^\beta} E^h(u^h) - C\left(\frac{1}{h^\beta} E^h\right)^{1/2} \geq -C$ dla $\beta = 2\alpha - 2$

Stąd wynika, że $\inf \frac{1}{h^\beta} J^h \in [-C, 0]$

Ponadto jeśli $\frac{1}{h^\beta} J^h(u^h) \leq C$, to wtedy $\frac{1}{h^\beta} E^h(u^h) \leq C$

<u>Twierdzenie 1:</u> Załóżmy, że $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ t. że $E^h(u^h) \leq C h^4$. Wtedy $\exists\ Q^h \in SO(3)$, $c^h \in \mathbb{R}^3$ t. że jeśli zdefiniujemy

$$y^h(x + t\vec{n}) = (Q^h)^T u^h\left(x + t\frac{h}{h_0}\vec{n}\right) - c^h \quad \in W^{1,2}(S^{h_0}, \mathbb{R}^3), \quad t \in \left(-\frac{h_0}{2}, \frac{h_0}{2}\right),$$

to mamy

(i) $y^h(x + t\vec{n}) \longrightarrow \pi(x + t\vec{n}) = x$ w $W^{1,2}(S^{h_0}, \mathbb{R}^3)$

(ii) $\underbrace{\frac{1}{h}\int_{-h_0/2}^{h_0/2} y^h(x+t\vec{n}) - x\, dt}_{V^h} \xrightarrow{W^{1,2}} V \in \mathcal{V}_1 := \{V \in W^{2,2}(S, \mathbb{R}^3) : \forall_{\tau,\eta \in T_x S}\ \underbrace{\langle \eta, \partial_\tau V\rangle + \langle \tau, \partial_\eta V\rangle}_{2\cdot \mathrm{sym}\nabla v} = 0\}$

$\mathcal{V}_1$ – inf. isometry of first order

(iii) $\frac{1}{h}\mathrm{sym}\nabla V^h \xrightarrow[L^2(S,\mathbb{R}^{2\times 2})]{} B \in \mathcal{B} := \{\lim_{L^2} \mathrm{sym}\nabla w^h,\ w^h \in W^{1,2}(S, \mathbb{R}^3)\}$

$\mathcal{B}$ – finite strain space

(iv) $$\liminf_{h\to 0} \frac{1}{h^4} E^h(u^h) \geq \mathcal{I}(V, B) = \overbrace{\frac{1}{2}\int_S Q_2\left(B - \tfrac{1}{2}(A^2)_{tan}\right)}^{\text{stretching term}} + \overbrace{\frac{1}{24}\int_S Q_2\left(\nabla(A\vec{n}) - A\Pi\right)}^{\text{bending term}}$$

<u>Fakt:</u> Jeśli $V \in \mathcal{V}_1$, to wtedy $\exists!\ A: S \to \mathbb{R}^{3\times 3}$ t. że dla p.w. $x \in S$ $A(x) \in so(3)$ oraz $\forall \tau \in T_x S \quad \partial_\tau V(x) = A(x)\tau$

Twierdzenie 2: $\forall V \in \mathcal{V}_1 \quad \forall B \in \mathcal{B} \quad \exists u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ t.że

(i) $y^h \to \pi$ w $W^{1,2}$

(ii) $V^h \to V$ w $W^{1,2}$

(iii) $\frac{1}{h} \operatorname{sym} \nabla V^h \to B$ w L^2

(iv) $\lim_{h\to 0} \frac{1}{h^4} E^h(u^h) = \mathcal{I}(V,B)$,

gdzie
$$\underset{\mathbb{R}^{3\times 3}}{Q_3(F)} = D^2 W(Id)(F,F)$$
$$\underset{\mathbb{R}^{2\times 2}}{Q_2(G)} = \min\{Q_3(F);\ F_{tan} = G\}$$

$$u^h|_S \simeq id + hV + h^2 \underset{\text{styczne}}{w_h}$$

Def: $S \subset \mathbb{R}^3$ - 2-wymiarowa powierzchnia. Definiujemy $\mathcal{V}_n$ - przestrzeń infinitezymalnych izometrii n-tego rzędu:

$$\mathcal{V}_n = \{(V_1, V_2, \dots, V_n):\ \forall V_i \in W^{2,2}(S,\mathbb{R}^3) \text{ t.że } \varphi^\varepsilon = id + \varepsilon V_1 + \varepsilon^2 V_2 + \dots + \varepsilon^n V_n$$
$$\forall \tau \in T_x S \quad |\partial_\tau \varphi^\varepsilon|^2 - |\tau|^2 = O(\varepsilon^{n+1})\}$$

Definiujemy exact isometries: $u \in W^{2,2}(S,\mathbb{R}^3)$: $(\nabla u)^T \nabla u = Id$

Fakt: $V \in \mathcal{V}_1$, $\tau \in T_x S$, $|\tau| = 1$: $|\partial_\tau(id + \varepsilon V)|^2 - 1 = \langle \tau + \varepsilon \partial_\tau V, \tau + \varepsilon \partial_\tau V\rangle - 1 =$

$$= 2\varepsilon \underbrace{\langle \partial_\tau V, \tau\rangle}_{=0} + O(\varepsilon^2)$$

$$\iff \operatorname{sym} \nabla V = 0,$$

czyli obie definicje $\mathcal{V}_1$ się pokrywają i dostajemy równoważny war

$$\mathcal{V}_1 = \{V \in W^{2,2}(S,\mathbb{R}^3):\ \forall \tau \in T_x S \quad \langle \tau, \partial_\tau V\rangle = 0\}$$

Fakt: $(V_1, V_2) \in \mathcal{V}_2 \iff \langle \tau + \varepsilon \partial_\tau V_1 + \varepsilon^2 \partial_\tau V_2, \tau + \varepsilon \partial_\tau V_1 + \varepsilon^2 \partial_\tau V_2\rangle - 1 =$

$$= 2\varepsilon \langle \partial_\tau V, \tau\rangle + \varepsilon^2 (2\langle \partial_\tau V_2, \tau\rangle + \langle \partial_\tau V_1, \partial_\tau V_1\rangle) + O(\varepsilon^3)$$

$$= [2 \operatorname{sym} \nabla V_2 - (A^2)_{tan}](\tau, \tau) = 0$$

Uwaga: Rozważmy deformację $\varphi^h(x) = x + hV(x) + h^2 w^h : S \to \mathbb{R}^3$, gdzie $\mathrm{sym}\nabla w_h \to B$

i policzmy zmianę metryki : $|\partial_\tau \varphi^h|^2 - |\tau|^2 = h^2 \cdot (2\underbrace{\mathrm{sym}\nabla w_h}_{\to B} - (A^2)_{tan}) + O(h^3)$

Zadanie: Pokazać, że wektor normalny do powierzchni $\varphi^h(S)$

$\vec{n}^h = \underset{\substack{\uparrow \\ \text{w. normalny do } S}}{\vec{n}} + h \cdot A\vec{n} + O(h^2)$

$\underset{\substack{\shortparallel \\ \nabla\vec{n}^h}}{\Pi^h} - \underset{\substack{\uparrow \\ \nabla\vec{n}}}{\Pi} = h\,(\nabla(A\vec{n}) - A\Pi)_{tan} + O(h^2)$

Przykład: $S \subset \mathbb{R}^2$ ① $\mathcal{V}_1 \ni V = (V^1, V^2, V^3) \iff 0 = \cancel{\mathrm{sym}\nabla V} = \mathrm{sym}\nabla(V^1, V^2) \iff$

$\iff V_{tan} = Ax + b, \quad A \in so(2)$

„Modulo rigid motion" $V \ni \mathcal{V}_1 \iff V = ve_3$

② $B \in \mathcal{B} \iff B \xleftarrow{L^2} \mathrm{sym}\nabla w_h$, $w_h \in W^{1,2}(S, \mathbb{R}^2)$, to wtedy $\|\mathrm{sym}\nabla w_h\|_{L_2} \leq C$

KORN $\Rightarrow \|\underbrace{w_h - (A^h x + c^h)}_{\tilde{w}_h \in so(2)}\|_{W^{1,2}} \leq C$

[Zad. $S \subset \mathbb{R}^3$ powierzchnia

$V \in W^{1,2}(S, \mathbb{R}^3)$, $V = V_{tan} + (V\vec{n})\vec{n}$, $\mathrm{sym}\nabla V = \mathrm{sym}\nabla V_{tan} + (V\vec{n})\Pi$]

$\Rightarrow \tilde{w}_h \xrightarrow{W^{1,2}} w$

$\mathrm{sym}\nabla w_h = \mathrm{sym}\nabla\tilde{w}_h \xrightarrow{L_2} \mathrm{sym}\nabla w = B$

$\Rightarrow \underline{\mathcal{B} = \{\mathrm{sym}\nabla w : w \in W^{1,2}(S, \mathbb{R}^2)\}}$.

③ $V = (0,0,v) \longrightarrow A = \left[\begin{array}{cc|c} 0 & 0 & -\partial_1 v \\ 0 & 0 & -\partial_2 v \\ \partial_1 v & \partial_2 v & 0 \end{array}\right]$, $A^2 = \begin{bmatrix} -\nabla v \otimes \nabla v & 0 \\ 0 & -|\nabla v|^2 \end{bmatrix}$

$(A^2)_{tan} = -\nabla v \otimes \nabla v$

$\cancel{B_1}$ $B - \frac{1}{2}(A^2)_{tan} = \underset{\substack{\cap \\ W^{1,2}(S,\mathbb{R}^2)}}{\mathrm{sym}\nabla w} + \frac{1}{2}\nabla v \otimes \nabla v$

$(\nabla(A\vec{n}) - A\Pi)_{tan} = -\nabla^2 v$

$$\mathcal{I}(v,B) = \mathcal{I}(v, w) = \frac{1}{2}\int_S Q_2(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v) + \frac{1}{24}\int_S Q_2(\nabla^2 v)$$

v — out-of-plane displacement; w — in-plane displacement

$\boxed{\text{von Karman functional}}$

$S \subseteq \mathbb{R}^2$

[LINEAR ELASTICITY]

Twierdzenie: $(\beta > 4)$ ($\Leftrightarrow$ $(\alpha > 3)$): Niech $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ t.że $E^h(u^h) \leq Ch^\beta$. wtedy $\exists Q^h, c^h$ t.że dla y^h mamy:

(i) $y^h \longrightarrow (x', 0)$ w $W^{1,2}(S^{h_0}, \mathbb{R}^3)$

(ii) $v^h = \frac{1}{h^{\frac{\beta}{2}-1}} \int_{-h_0/2}^{h_0/2} (y^h)_3 \, dx_3 \xrightarrow{W^{1,2}} v \in W^{2,2}(S, \mathbb{R})$

(iii) $\frac{1}{h^\beta} E^h(u^h) \xrightarrow{\Gamma} \mathcal{I}_{lin}(v) = \frac{1}{24}\int_S Q_2(\nabla^2 v)$ $\quad [u^h|_S = id + h^{\frac{\beta}{2}}\begin{pmatrix}0\\0\\v\end{pmatrix} + o(h^{\beta/2})]$

Twierdzenie 2 $(\beta = 4)$, $E^h(u^h) \leq Ch^4$. wtedy [VON KARMAN THEORY]

(i) $y^h \xrightarrow{W^{1,2}} (x', 0)$

(ii) $v^h = \frac{1}{h} \int_{-h_0/2}^{h_0/2} (y^h)_3 \, dx_3 \xrightarrow{W^{1,2}} v \in W^{2,2}(S, \mathbb{R})$

(iii) $\frac{1}{h^2} \cancel{\mathrm{sym}\nabla_{tan} y^h} \left(\int_{-h_0/2}^{h_0/2} (y^h)_{tan} \, dx^3 - x' \right) \xrightarrow{W^{1,2}} w \in W^{1,2}(S, \mathbb{R}^2)$

(iv) $\frac{1}{h^4} E^h(u^h) \xrightarrow{\Gamma} \mathcal{I}_4(v,w) = \frac{1}{2}\int_S Q_2(\mathrm{sym}\nabla w + \frac{1}{2}\nabla w \otimes \nabla w) + \frac{1}{24}\int_S Q_2(\nabla^2 v)$

[LINEAR KIRCHHOFF]

Twierdzenie 3: $2 < \beta < 4$ $\quad E^h(u^h) \leq Ch^\beta$. Wtedy $\exists Q^h, c^h$ t.że

(i) $y^h \to (x', 0)$ w $W^{1,2}$

(ii) $v^h = \frac{1}{h^{\beta/2 - 1}} \int_{-h_0/2}^{h_0/2} (y^h)_3 \, dx_3 \xrightarrow{W^{1,2}} v \in W^{2,2}(S, \mathbb{R})$ i $\det \nabla^2 v = 0$

(iii) $\frac{1}{h^\beta} E^h(u^h) \xrightarrow{\Gamma} \mathcal{I}_{linkir}(v) = \frac{1}{24}\int_S Q_2(\nabla^2 v)$

Twierdzenie 4: $\beta = 2$ [KIRCHHOFF]. $E^h(u^h) \le C \cdot h^2$. wtedy $\exists\, Q^h, c^h$ t.że

(i) $y^h \xrightarrow[W^{1,2}(S^{h_0}, \mathbb{R}^3)]{} y\pi$, gdzie $y \in W^{2,2}(S, \mathbb{R}^3)$, $(\nabla y)^T \nabla y = Id$

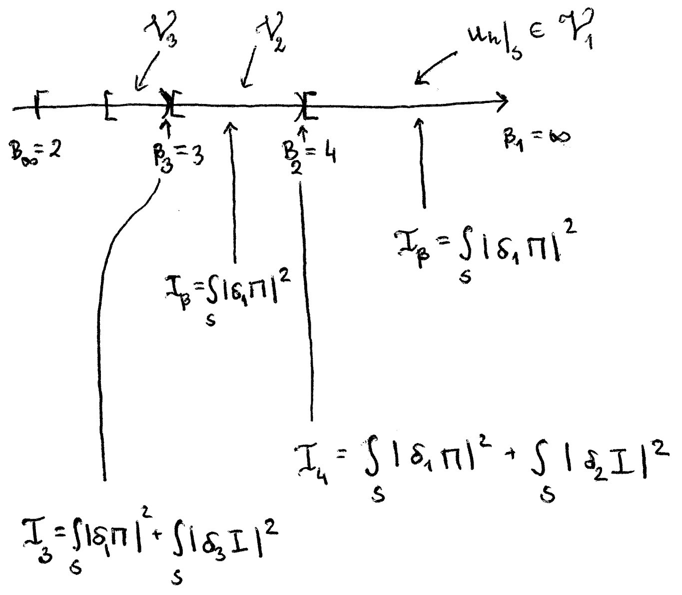
(ii) $\frac{1}{h^2} E^h(u^h) \xrightarrow{\Gamma} I_{Kir}(y) = \frac{1}{24} \int_S Q_2(\Pi(y) - \Pi)$

$\alpha(1)$

$\underset{y(S)}{=}$ forma podstawowa S

Uwaga: Powyższe twierdzenie (4) zachodzi dla powierzchni $S \subset \mathbb{R}^3$.
Inna nazwa powyższego tw, to PURE BENDING

S - 2d powierzchnia. Definiujemy $\beta_i = 2 + \frac{2}{i-1}$



Hierarchy of shell energies

Uwaga: $S \subset \mathbb{R}^2$, $(V_1, V_2) \in \mathcal{V}_2 \Rightarrow V_1 \in \mathcal{V}_1 \Leftrightarrow V_1 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$

$\underset{\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}}{\shortparallel} \quad \underset{\begin{pmatrix} w \\ 0 \end{pmatrix}}{\shortparallel}$ $\qquad \mathrm{sym}\nabla w + \frac{1}{2} \nabla v \otimes \nabla v = 0$

$$|\partial_\tau (id + \varepsilon v e_3 + \varepsilon^2 w)|^2 - 1 = O(h^3)$$
$$|\tau + \varepsilon(\partial_\tau v) e_3 + \varepsilon^2 \partial_\tau w|^2 - 1 = \varepsilon^2 ((\nabla v \otimes \nabla v)(\tau, \tau) + 2(\mathrm{sym}\nabla w)(\tau, \tau)) + O(\varepsilon^3)$$

$$\exists\, w : \mathrm{sym}\nabla w + \frac{1}{2} \nabla v \otimes \nabla v = 0 \Leftrightarrow \nabla v \otimes \nabla v = \mathrm{sym}\nabla \tilde{w} \Leftrightarrow \mathrm{rot}^T \mathrm{rot}\,(\nabla w \otimes \nabla w) \overset{\Uparrow}{=} \frac{1}{2} \det \nabla^2 v$$

$\mathbb{R}^{2,(3)} \quad (3\times 3)$

Fakt: Dane jest $B \in L^2(\breve{S}, \mathbb{R}^{2\times 2})$. wtedy następujące warunki są równoważne

(i) $\exists\, w \quad B = \mathrm{sym}\nabla w$

(ii) $\mathrm{rot}^T \mathrm{rot}\, B = 0$

Twierdzenie: Niech $v \in W^{2,2}(S,\mathbb{R}) \cap W^{1,\infty}$ t.że $\det \nabla^2 v = 0$. wtedy

$\exists\, w_\varepsilon \in W^{2,2}(S,\mathbb{R}^2)$, $\|w_\varepsilon\|_{W^{2,2}} \le C$ $\forall_\varepsilon$ $\varphi_\varepsilon = id + \varepsilon v e_3 + \varepsilon^2 w_\varepsilon$ jest izometrią

$\varphi_\varepsilon \in W^{2,2}(S,\mathbb{R}^3)$, $(\nabla\varphi_\varepsilon)^T(\nabla\varphi_\varepsilon) = Id$

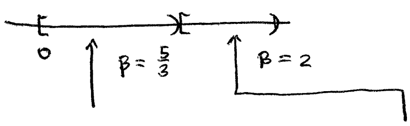
[MATCHING PROPERTY]

[S. Müller, Pakzad]

Twierdzenie: (M.L., Mora, Pakzad)

Niech $V \in W^{2,2} \cap C^{2,\alpha}(S,\mathbb{R}^3) \cap \mathcal{V}_1$, gdzie S - 2d powierzchnia, $S \in C^{2,\alpha}$, S-ściśle wypukła (tj $\langle \Pi(x)\tau, \tau \rangle \geq c|\tau|^2$). Wówczas istnieje ciąg $w_\varepsilon \in C^{2,\alpha}(S,\mathbb{R}^3)$, t.że $\|w_\varepsilon\|_{C^{2,\alpha}} \le C$ $\forall_\varepsilon$ $id + \varepsilon V + \varepsilon^2 w_\varepsilon$ jest izometrią

Co dla $\beta < 2$?



membrane theory
LeDret - Raoult

? Conjecture
S. Conti
F. Maggi
S. Venkataramani

Wielkie Pytanie: $\beta = \frac{5}{3}$
related to paper crumpling / origami

W razie problemów: Lewicka@pitt.edu