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DiBenedetto „Degenerate Parabolic Equations"

Preliminaria:

$S_\varepsilon : \mathcal{D}(\mathbb{R}^N) \to \mathcal{D}(\mathbb{R}^N)$ $\quad \varphi \mapsto \varphi * \omega_\varepsilon$

$\omega_\varepsilon(x) = \frac{1}{\varepsilon^N}\,\omega\left(\frac{x}{\varepsilon}\right)$

$S_\varepsilon : \mathcal{D}'(\mathbb{R}^N) \to \mathcal{D}'(\mathbb{R}^N)$ $\quad \omega \in \mathcal{D}(\mathbb{R}^N),\ \omega \geq 0$

$\langle S_\varepsilon(f), \varphi\rangle = \langle f, S_\varepsilon(\varphi)\rangle$ $\quad \operatorname{supp}\omega \subset B_1, \quad \int \omega = 1$

$\omega(x) = \tilde{\omega}(|x|)$

$S_\varepsilon(f) \in C^\infty(\mathbb{R}^N)$

$f \in L^p_{loc}(\mathbb{R}^N), \ 1 \leq p < \infty$

$\Rightarrow \quad S_\varepsilon(f) \in L^p_{loc}(\mathbb{R}^N)$

$S_\varepsilon(f) \to f$ w $L^p_{loc}(\mathbb{R}^N)$

Lemat Friedrichsa:

Niech $N \geq 2$, $\varrho \in L^\beta_{loc}(\mathbb{R}^N)$, $u \in (W^{1,q}_{loc}(\mathbb{R}^N))$,
gdzie $1 \leq q, \beta \leq \infty$, $\frac{1}{q} + \frac{1}{\beta} \leq 1$, $(q,\beta) \neq (1,\infty)$.

Niech $u \cdot \nabla\varrho = \operatorname{div}(\varrho u) - \varrho \operatorname{div} u \in \mathcal{D}'(\mathbb{R}^N)$.

Wówczas $S_\varepsilon(u\cdot\nabla\varrho) - u\cdot\nabla S_\varepsilon(\varrho) \to 0$ w $L^r_{loc}(\mathbb{R}^N)$,

gdzie
$$\begin{cases} r \in [1,q) & \text{jeżeli } \beta = \infty, \ q \in (1,\infty] \\ \frac{1}{\beta} + \frac{1}{q} \leq \frac{1}{r} \leq 1 & \text{w przeciwnym razie} \end{cases}$$

Dowód:

$$\langle S_\varepsilon(u\cdot\nabla\varrho), \varphi\rangle = \int_{\mathbb{R}^N} \varrho(y)u(y)\int_{\mathbb{R}^N} \nabla\omega_\varepsilon(x-y)\varphi(x)\,dx\,dy$$

$$- \int_{\mathbb{R}^N} \varrho(y)\operatorname{div}u(y)\int_{\mathbb{R}^N} \omega_\varepsilon(x-y)\varphi(x)\,dx\,dy$$

$$= \int_{\mathbb{R}^N} \varphi(x)\left(\int_{\mathbb{R}^N} \varrho(y)u(y)\cdot\nabla\omega_\varepsilon\right)$$

$$\langle u\cdot\nabla S_\varepsilon(\varrho); \varphi\rangle = \int_{\mathbb{R}^N} \varphi(x)\int_{\mathbb{R}^N} u(x)\cdot\nabla\omega_\varepsilon(x-y)\varrho(y)\,dy\,dx$$

$$\langle S_\varepsilon(u\cdot\nabla\rho) - u\cdot\nabla S_\varepsilon(\rho), \varphi\rangle = \underbrace{\int_{\mathbb{R}^N} \varphi(x)\left(\int_{\mathbb{R}^N}\rho(y)(u(y)-u(x))\cdot\nabla\omega_\varepsilon(x-y)dy\right.}_{I_\varepsilon}$$

$$\underbrace{-\int_{\mathbb{R}^N}\omega_\varepsilon(x-y)\rho(y)\,\mathrm{div}\,u(y))\,dx}_{J_\varepsilon}$$

$$J_\varepsilon = S_\varepsilon(\rho\,\mathrm{div}\,u) \to \rho\,\mathrm{div}\,u \quad w \quad L^r_{loc}(\mathbb{R}^N)$$

Stwierdzenie:

$$\|I_\varepsilon\|_{r_0, B_R} \le C(R,\tilde{\beta})\|\rho\|_{\tilde{\beta}, B_{R+1}} \|\nabla u\|_{q, B_{R+2}},$$

gdzie
$$\begin{cases} \frac{1}{r_0} = \frac{1}{\beta}+\frac{1}{q}, & \tilde{\beta}=\beta, \quad 1\le\beta<\infty \\ r_0\in[1,q), & \tilde{\beta} = \frac{r_0 q}{q-r_0} = \left(\frac{q}{r_0}\right)' r_0, \quad \text{jeśli} \quad \beta=\infty, q\neq\infty \\ r_0\in[1,\infty), & \tilde{\beta}=r_0, \quad \text{jeżeli} \quad \beta=q=\infty \end{cases}$$

$$\|I_\varepsilon\|^{r_0}_{r_0,B_R} = \int_{B_R}\left|\int_{|x-y|\le\varepsilon}\rho(y)(u(y)-u(x))\frac{1}{\varepsilon^{N+1}}\nabla\omega\left(\frac{x-y}{\varepsilon}\right)dy\right|^{r_0}dx =$$

$$= \left[\eta = \frac{x-y}{\varepsilon}\right] = \int_{B_R}\left|\int_{|\eta|\le 1}\rho(x-\varepsilon\eta)\frac{u(x-\varepsilon\eta)-u(x)}{\varepsilon}\nabla\omega(\eta)d\eta\right|^{r_0}dx \le$$

$$\le \int_{B_R}\left(\int_{|\eta|\le 1}\left|\rho(x-\varepsilon\eta)\frac{u(x-\varepsilon\eta)-u(x)}{\varepsilon}\right|^{r_0}\right)\left(\int_{|\eta|\le 1}|\nabla\omega(\eta)|^{r_0'}\right)^{\frac{r_0}{r_0'}}dx\Big)$$

$$= C\cdot\int_{|\eta|\le 1}\int_{x\in B_R}\left|\rho(x-\varepsilon\eta)\frac{u(x-\varepsilon\eta)-u(x)}{\varepsilon}\right|^{r_0}dx\,d\eta =$$

$$= [\xi = x-\varepsilon\eta] = C\cdot\int_{|\eta|\le 1}\int_{\xi\in B_R-\varepsilon\eta}|\rho(\xi)|^{r_0}\left|\frac{u(\xi+\varepsilon\eta)-u(\xi)}{\varepsilon}\right|^{r_0}d\xi\,d\eta \le$$

$$\le C\int_{B_{R+1}}\int_{|\eta|\le 1}|\rho(\xi)|^{r_0}\left(\int_{|\eta|\le 1}\left|\frac{u(\xi+\varepsilon\eta)-u(\xi)}{\varepsilon}\right|^q d\eta\right)^{\frac{r_0}{q}}d\xi \le$$

$$\le C\left(\int_{B_{R+1}}\int_{|\eta|\le 1}\left|\frac{u(\xi+\varepsilon\eta)-u(\xi)}{\varepsilon}\right|^q d\eta\,d\xi\right)^{\frac{r_0}{q}}\left(\int_{B_{R+1}}|\rho|^{\tilde{\beta}}d\xi\right)^{\frac{r_0}{\tilde{\beta}}}$$

Fakcik: $u(\xi+\varepsilon z) - u(\xi) = \varepsilon z\cdot\int_0^1\nabla u(\xi+t\varepsilon z)dt$ p.w.

$$\int_{B_{R+1}} \int_{|\eta|\le 1} \left| \frac{u(\xi+\varepsilon\eta)-u(\xi)}{\varepsilon} \right|^q d\eta\, d\xi \le$$

$$\le C \cdot \int_{B_{R+1}} \int_{|\eta|\le 1} \int_0^1 |\nabla u(\xi + t\varepsilon\eta)|^q dt =$$

$$= C \cdot \int_{|z|\le 1} \int_{t\in[0,1]} \int_{\eta \in B_{R+1}} |\nabla u(\eta)|^q d\eta\, dt\, dz$$

$$\le C \cdot \int_{B_{R+2}} |\nabla u(\eta)|^q d\eta \qquad \blacksquare$$

Chcemy pokazać

$(*)$ $\quad I_\varepsilon \to \varrho \operatorname{div} u \quad$ w $\quad L^r_{loc}(\mathbb{R}^n)$

Stwierdzenie: wystarczy pokazać $(*)$ przy założeniu $\varrho \in C_0^\infty(\mathbb{R}^N)$

Niech $\varrho_n \in C_0^\infty(\mathbb{R}^N)$ $\quad \varrho_n \to \varrho \quad$ w $\quad L^\beta(B_{R+1})$

$$\| I_\varepsilon - \varrho \operatorname{div} u \|_{r_0, B_{R+1}} \le \left\| \int_{\mathbb{R}^N} (\varrho(y) - \varrho_n(y))(u(y) - u(\cdot)) \cdot \nabla \omega_\varepsilon(\cdot - y) \right\|_{r_0, B_{R+1}}$$

$$+ \left\| \int_{\mathbb{R}^N} \varrho_n(y)(u(y) - u(\cdot)) \cdot \nabla \omega_\varepsilon(\cdot - y)\, dy - \varrho_n \operatorname{div} u \right\|_{r_0, B_{R+1}} +$$

$$+ \| (\varrho_n - \varrho) \operatorname{div} u \|_{r_0, B_{R+1}}.$$

$$I_\varepsilon(x) = \left[z = \cancel{\tfrac{x-y}{\varepsilon}} \ \frac{y-x}{\varepsilon} \right] = - \int_{|z|\le 1} \varrho(x+\varepsilon z) \frac{u(x+\varepsilon z) - u(x)}{\varepsilon} \nabla \omega(z)\, dz$$

$$\varrho(x+\varepsilon z) \to \varrho(x) \qquad \forall (x,z) \in B_{R+1} \times B_1$$

$$\frac{u(x+\varepsilon z) - u(x)}{\varepsilon} = z \cdot \int_0^1 \nabla u(x + \varepsilon t z)\, dt \xrightarrow{\varepsilon \to 0} z \cdot \nabla u(x) \quad \text{dla p.w. } (x,z)$$

$$\int_{\mathbb{R}^N} I_\varepsilon \varphi \to - \sum_{i,j} \left(\int_{B_1} z_i \partial_j \omega(z)\, dz \right) \cdot \left(\int_{\mathbb{R}^N} \varrho(x) \varphi(x) \partial_i u_j(x)\, dx \right)$$

$$\left(\text{bo} = - \int_{B_1} \int_{\mathbb{R}^N} \varrho(x) (z \cdot \nabla) u(x) \cdot \nabla \omega(z)\, dx\, dz \right)$$

$$= \int_{\mathbb{R}^N} \varrho \operatorname{div} u \varphi.$$