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Tw: $u_0 \in L_2(\Omega)$, $\operatorname{div} u_0 = 0$ w $\mathcal{D}'(\Omega)$, $n = 2, 3$. wtedy istnieje co najmniej 1 słabe rozwiązanie NS globalne w czasie. spełniające

$$\|u\|_{L_\infty(0,T;\, L_2(\Omega))} + \int_0^T \|\nabla u\|^2_{L_2(\Omega)}\, dt \leq C \|u_t\|_{L_2(\Omega)}$$

$$\Rightarrow \|u(T)\|^2_{L_2(\Omega)} + 2\nu \int_0^T \|\nabla u\|^2_{L_2(\Omega)}\, dt \leq \|u(0)\|^2_{L_2(\Omega)}$$

$$0 \leq s_0 < s_1 \leq T$$

$$\|u(s_1)\|^2_{L_2(\Omega)} + 2\nu \int_{s_0}^{s_1} \|\nabla u\|^2_{L_2(\Omega)}\, dt \leq \|u(s_0)\|^2_{L_2(\Omega)}$$

To można uzyskać z aproksymacji, na początku dla liczb wymiernych.

$$-\int_0^T \int_\Omega u \varphi_t \, dx dt + \int_0^T \int_\Omega u \nabla u \, \phi \, dx dt + \nu \int_0^T \int_\Omega \nabla u : \nabla \varphi \, dx = \int_\Omega u_0 \varphi(x,0)\, dx$$

$$\varphi \in \mathcal{D}(\Omega), \quad \operatorname{div} \varphi = 0, \quad \varphi = 0 |_{\partial\Omega}$$

Ten warunek sprawia problemy, bo dla funkcji lokalizującej η $\operatorname{div}(\varphi \cdot \eta) = 0$.

Chcemy odzyskać ciśnienie

Ciśnienie:

Stw: (de Rham) $f \in \mathcal{D}'(\Omega, \mathbb{R}^n)$

$$f = \nabla p \iff (f, v) = 0 \quad \text{dla} \quad v \in \mathcal{D}(\Omega),\ \operatorname{div} v = 0. \qquad (KK)$$

Dla Ω - ograniczonego zachodzą nierówności

$$\|p\|_{L_2(\Omega)/\mathbb{R}} \leq C(\Omega) \|\nabla p\|_{L_2(\Omega)} \qquad \left(p \in L_2(\Omega)/\mathbb{R} \Rightarrow \int_\Omega p = 0\right)$$

$$\|p\|_{L_2(\Omega)/\mathbb{R}} \leq C(\Omega) \|\nabla p\|_{H^{-1}(\Omega)}$$

Druga zachodzi, bo

$$\|\nabla p\|_{H^{-1}} = \sup \int_\Omega \nabla p \cdot \phi \, dx = \sup - \int_\Omega p \cdot \operatorname{div} \phi \, dx.$$

$$\phi \in H^1_0(\Omega, \mathbb{R}^n)$$

$$f \in L_2(\Omega)/\mathbb{R}$$

$$\phi: \ \operatorname{div} \phi = f, \quad \phi = 0$$

$\sup_{\Phi_A} \int \nabla p \, \Phi_A = \sup_{\Phi_B} \int \nabla p \cdot \Phi_B$ – chcemy pokazać taką równość („$\geq$" jest)

$\Phi_A \in H_A = \{ H^1(\Omega, \mathbb{R}^n) : \Phi \cdot n = 0 \text{ na } \partial\Omega \}$

$\phi_B \in H_B = \{ H^1(\Omega, \mathbb{R}^n) : \phi = 0 \text{ na } \partial\Omega \}$

$H_A \supset H_B$

$\phi_A - \phi_B = \phi_c$

Dla każdej funkcji ϕ_A znajdziemy ϕ_B :

ϕ_c : $\operatorname{div} \phi_c = 0$, $\phi_c \cdot n = 0$

$\int_\Omega \nabla p \cdot \phi_c \, dx = - \int_\Omega p \operatorname{div} \phi_c \, dx + \int_{\partial\Omega} p \phi_c \cdot n \, d\sigma = 0$

Gdybyśmy to wiedzieli byłoby ok

$(v_t + v\nabla v - \mu \Delta v, \phi) = 0 \quad / \int_0^T dt$

$\left(\underset{L_2(\Omega)}{v(T) - v(0)} + \underset{H^{-1}(\Omega)}{\int_0^T v \nabla v \, dt} - \underset{H^{-1}(\Omega)}{\mu \Delta \int_0^T v \, dt}, \phi \right) = 0$

$\nabla p = v(T) - v_0 + \int_0^T v \nabla v \, dt - \mu \Delta \int_0^T v \, dt$

$p = P_T$

$\nabla p = v_t + v\nabla v - \mu \Delta v \quad w \quad \mathcal{D}'(\Omega)$ <u>$d=2$</u>

<u>Równanie dwuwymiarowe</u>:

Zauważmy, że słabe rozwiązania NS w 2-dim spełniają

$\int_\Omega u_t \phi \, dx + \int_\Omega u \cdot \nabla u \, \phi \, dx + \nu \int_\Omega \nabla u \nabla \varphi \, dx = 0$

$\varphi \in \mathcal{D}(\Omega)$, $\operatorname{div} \phi = 0$, $\phi|_{\partial\Omega} = 0$

$\nabla u \in L_2(0,T; L_2(\Omega))$

$u\nabla u \in L_2(0,T; H^{-1}(\Omega))$

$\Delta v \in L_2(0,T, H^{-1}(\Omega))$

czyli możemy testować $L_2(0,T, H^1(\Omega))$

Lemat Ładyżeńskiej (Ладыженская) [Ladyzhenskaya]

$u \in H_0^1(\Omega)$

$$\|u\|_{L_4(\Omega)} \le C \cdot \|u\|_{L_2}^{1/2} \cdot \|\nabla u\|_{L_2(\Omega)}^{1/2}$$

Tw: Niech v^1, v^2 będą dwoma słabymi rozwiązaniami NS dla tych samych danych początkowych $v_0 \in L_2(\Omega)$, wtedy

$$v^1 \equiv v^2 \qquad \dim \Omega = 2 .$$

Dowód:

$i = 1, 2$

$$(v_t^i, \phi)_{L_2} + (v^i \nabla v^i, \phi) + \mu(\nabla v^i, \phi) = 0$$

$$v^i|_{t=0} = v_0$$

$-$

$$((v^1 - v^2)_t, \phi) + (v^1 \nabla (v^1 - v^2), \phi) + \mu(\nabla(v^1 - v^2), \nabla \varphi) =$$

$$= -((v^1 - v^2)\nabla v^2, \phi) \qquad , \quad \text{kładziemy } \phi = v^1 - v^2$$

$$v^1 - v^2|_{t=0} = 0$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v^1 - v^2|^2 dx + \mu \int_\Omega |\nabla(v^1 - v^2)|^2 dx \le \int_\Omega |\nabla v^2| (v^1 - v^2)^2 dx$$

dla każdego $T > 0$

$$\sup_{0 \le t < T} \|v^1 - v^2\|^2_{L_2(\Omega)} + \|\nabla(v^1 - v^2)\|^2_{L_2(0,T; L_2(\Omega))} \le \int_0^T \int_\Omega |\nabla v^2| \, |v^1 - v^2| \, dx$$

$$\|v^1 - v^2\|^2_{L_4(\Omega \times (0,t))} \le C \left[\sup_t \int_\Omega |v^1 - v^2|^2 dx + \int_0^t \int_\Omega |\nabla(v^1 - v^2)|^2 dx dt \right]$$

$$\int_0^T \|f\|^4_{L_4} dt \le C \cdot \int_0^T \|f\|^2_{L_2} \|\nabla f\|^2_{L_2} dt \le C \sup_t \|f\|^2_{L_2} \int_0^T \|\nabla f\|^2_{L_2} dt$$

$$\le C\left(\sup_t \|f\|^2_{L_2}\right)^2 + \left(\int_0^t \|\nabla u\|^2_{L_2}\right)^2$$

$$\sup_t \int (v^1 - v^2)^2 dx + \int_0^t \int |\nabla(v^1 - v^2)|^2 dx dt \le \left(\int_0^t \int_\Omega |\nabla u^2|^2 dx dt\right)^{1/2} \cdot$$

$$\left(\int_0^t \int_\Omega |v^1 - v^2|^4 dx dt\right)^{1/2} \le \left(\int_0^t \int_\Omega |\nabla u^2|^2 dx dt\right)^{1/2} C \left[\sup_t \int (v^1 - v^2)^2 dx\right.$$

$$\left. + \int_0^t \int_\Omega |\nabla(v^1 - v^2)|^2 dx dt \right)$$

$$\int_0^t \int_\Omega |\nabla v^2|^2 dx dt << 1$$

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2d w $\mathbb{R}^2$ - przypadek szczególny

Równanie N-S:

$$\left.\begin{aligned} v_t + v\nabla v - \mu \Delta v + \nabla p &= 0 \\ \operatorname{div} v &= 0 \end{aligned}\right\} \mathbb{R}^2 \times (0,T)$$

$$v|_{t=0} = v_0$$

Rotacja w $\mathbb{R}^3$

$(\operatorname{rot} v)^{(k)} = \varepsilon_{ijk} \partial x_i v^j$, gdzie $\varepsilon_{ijk} = 0$, gdy któreś się powtarzają

$= 1$, gdy permutacja jest parzysta

$= -1$, gdy nieparzysta

Co się dzieje dla $\mathbb{R}^2$?

$\operatorname{rot}(v\nabla v) = [v^i \partial_i v^2]_1 - [v^i \partial x_i v^1]_{12} = v\nabla(\operatorname{rot} v) + v^i_1 \partial_i v^2 - v^i_2 \partial_i v^1 = 0$

Natomiast w 3d:

$\operatorname{rot}(v\nabla v) = v\nabla(\operatorname{rot} v) - \operatorname{rot} v \nabla v$

Bierzemy rotację w 2d w NS:

$$\omega_t + v\nabla\omega - \mu\Delta\omega = 0 \quad \text{w } \mathbb{R}^2$$

$$\operatorname{rot} v = \omega \qquad v = K(\omega)$$

$$\operatorname{div} v = 0$$

$$\omega|_{t=0} = \omega_0 \in L^1(\mathbb{R}^2) \cap L_\infty(\mathbb{R}^2)$$

Cel: Istnienie, jednoznaczność

$\omega \in L_\infty(0,T; L_1 \cap L_\infty(\mathbb{R}^2))$

$\operatorname{div} = 0$

funkcja prądu (potoku)

$v = (-\varphi_{x_2}, \varphi_{x_1}) = \nabla^+ \varphi$

$\operatorname{rot}(-\varphi_{x_2}, \varphi_{x_1}) = \Delta\varphi$

$\Delta\varphi = \omega$ w $\mathbb{R}^2$

$$\varphi = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y|\, \omega(y)\,dy \qquad \text{Biot-Savart}$$

$$\nabla\varphi = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\, \omega(y)\,dy$$

Próba znalezienia oszacowań a priori. Zakładamy, że wszystko jest gładkie.

$\{w>0\}$ Uwaga: Nie zawsze się tak da, ale i tak to zrobimy, bo wiemy, że istnieje $\sigma_n \to 0$, $\{w-\sigma_n\}$ jest regularny.

całkujemy po zbiorze $\{w>0\}$

$$\frac{d}{dt}\int_{\{w>0\}} w\,dx + \int_{\{w>0\}} \overset{=0}{v\cdot\nabla w} - \mu\cdot\overset{\geq 0}{\int_{\{w>0\}}\Delta w\,dx} = 0$$

- $\frac{d}{dt}\int_{\{w>0\}} w\,dx = \int_{\mathbb{R}^2}\partial_t(w_t)\,dx$
- $\int_{\{w>0\}} v\cdot\nabla w = -\int_{\{w>0\}} \overset{0}{\overset{\|}{\operatorname{div} v}}\cdot w\,dx + \int_{\partial\{w>0\}} n\cdot v\cdot \overset{0}{\overset{\|}{w}}\,d\sigma$
- $-\mu\int_{\{w>0\}}\Delta w\,d\sigma = -\mu\int_{\partial\{w>0\}}\frac{\partial w}{\partial n}\,d\sigma \geq 0$

$w = w_+ - w_-$

$\int w_-\,dx \leq \int w_{0-}\,dx$

$\frac{d}{dt}\int_{\mathbb{R}^2} w_+\,dx \leq 0$ $\qquad \int_{\mathbb{R}^2} w_+\,dx \leq \int_{\mathbb{R}^2} w_{0+}\,dx$

Uwaga: Gdy $\|u\|_p \leq$ wspólnie ogr dla $p\in(0,\infty)$ $\Rightarrow$ $u\in L_\infty$

$\|w\|_{L_\infty(0,T;\,L_1(\mathbb{R}^2))} \leq \|w_0\|_{L_1(\mathbb{R}^2)}$

Testujemy $|w|^{p-2}w$

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|w|^p\,dx + \underbrace{\frac{1}{p}\int_{\mathbb{R}^2} v\nabla|w|^p dx}_{=0} - \mu\int_{\mathbb{R}^2}\Delta w\,|w|^{p-2}w\,dx = 0$$

$$\frac{d}{dt}\frac{1}{p}\int_{\mathbb{R}^2}|w|^p dx + \underbrace{\mu(p-1)\int_{\mathbb{R}^2}|\nabla w|^2|w|^{p-2}dx}_{>0} = 0$$

Metoda Mosera - lepiej na ogr.

$$\Rightarrow \|w\|_{L_p(\mathbb{R}^2)} \leq \|w_0\|_{L_p(\mathbb{R}^2)} \leq C(\|w_0\|_{L_1} + \|w_0\|_{L_\infty})$$

Testujemy $(\omega-k)_+$

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^2}(\omega-k)_+^2 + \frac{1}{2}\overset{0}{\overset{\parallel}{\int v\nabla(\omega-k)_+^2}} + \mu\int|\nabla(\omega-k)_+|^2\,dx = 0$$

$\inf \omega_0 < \omega \leq \sup \omega_0$

$$\sup_t \int_{\mathbb{R}^2}(\omega-k)_+^2 \leq \int_{\mathbb{R}^2}(\omega_0-k)_+^2\,dx = 0$$

<u>Istnienie:</u> $\operatorname{div} v = 0$ istnieje funkcja prądu $v = (-\partial_{x_2}\varphi, \partial_{x_1}\varphi)$

$\operatorname{rot} v = \omega$

$\Delta\varphi = \omega$ w $\mathbb{R}^2$ $\qquad \nabla^\perp\varphi = v$

$\omega \in L_1 \cap L_\infty(\mathbb{R}^2)$ $\qquad \omega \in L_p,\ p \in [1,\infty]$

$\|\nabla^2\varphi\|_{L_p} \leq c\|\omega\|_{L_p}$ $\qquad$ tw. Calderona - Zygmunta

$C(p) \sim p, \frac{1}{p-1}$

zatem z tw. Sobolewa

$\|\nabla\varphi\|_{L_{\frac{2p}{2-p}}} \leq c\|\omega\|_{L_p}$, $\quad p \in (1,2)$

$v \in L_{2+\sigma} \cap W^1_{\infty-\sigma}(\mathbb{R}^2)$, gdzie $\infty-\sigma$ - dowolnie duża liczba, mniejsza od ∞.

$\nabla v \in BMO(\mathbb{R}^2)$ - dla chętnych Stein gruby

Chcemy skonstruować $K: X \to X$ t.że $v \mapsto \tilde{v}$

$\omega_t + v\nabla\omega - \mu\Delta\omega = 0$ $\qquad B_R \times (0,T)$

$\omega = 0$

$\omega|_{t=0} = \omega_0$

$v \in L_\infty(0,T; L_{2+\sigma} \cap W^1_{\infty-\sigma}(\mathbb{R}^2))$

$v = v|_{B_R} \Rightarrow \omega_R:\ \omega_t + v\nabla\omega - \Delta\omega = 0$

$\omega|_{\partial B_R} = 0$ w B_R

$\omega_R \xrightarrow{R\to\infty} \omega_\infty$

$\tilde{v} = K_{BS}\,\omega_\infty$; nasze pnie $v \mapsto \tilde{v}$

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$K : X \to X \qquad v \to \tilde{v}$

$w_t + v\cdot\nabla w - \mu\cdot\Delta w = 0$

$w|_{t=0} = w_0$

$\mathrm{rot}\,\tilde{v} = w$

$\mathrm{div}\,\tilde{v} = 0$

$w_R\,, \quad R \to \infty$

$v \in L_\infty(0,T; L_{2+\sigma} \cap W^1_{\infty-\sigma}(\mathbb{R}^2))$

$B_R \subset \mathbb{R}^2 \qquad w_t^R + v\nabla w^R - \mu\Delta w^r = 0 \quad , \quad w_R = 0 \quad$ na $\quad \partial B_R$

$w \sim \sum a_j^N(t) w_j(x)$

$w_R \in L_\infty(0,T; L_2(B_R)) \cap L_2(0,T; H_0^1(B_R))$

$\tilde{v}(R) = K * E_w \qquad$ B-S

Jedynosć : $\quad w_1 - w_0 \qquad \tilde{v}^1 = K * Ew^1 \qquad \tilde{v}^2 = K * Ew^2$

$Ew = \begin{cases} w & x \in B_R \\ 0 & \text{w.p.p.} \end{cases}$ - rozszerzenie

$(w_1 - w_0)_t + v_1\nabla(w_1 - w_0) - \mu\Delta(w_1 - w_0) = (v_1 - v_0)\nabla w_1$

$\frac{1}{2}\frac{d}{dt}\int_{B_R}(w_1 - w_0)^2 dx + \mu\int_{B_R}|\nabla(w_1 - w_0)|^2 dx \le \int_{B_R}|v_1 - v_0||\nabla(w_1 - w_0)||w_1|$

$\le \frac{\mu}{2}\int_{B_R}|\nabla(w_1 - w_0)|^2 dx + c_\mu\int_{B_R}(v_1 - v_0)^2|w_1|^2$

$w_1 \in L_\infty(0,T; L_\infty(B_R))$

$\sup_{t<T}\|w_1 - w_0\|^2_{L_2(B_R)} \le c_\mu(|w_1|_{L_\infty})\int_0^T\|v_1 - v_0\|^2_{L_2(B_R)} dt$

$\tilde{v}_i : \quad \mathrm{rot}\,\tilde{v}_i = w_i \in L_1 \cap L_2$

$\mathrm{div}\,\tilde{v}_i = 0$

$\Delta\varphi_i = w_i \qquad \tilde{v}_i = \nabla^\perp\varphi$

$\nabla\varphi_i \in L_{2+\sigma}(\mathbb{R}^2)$

$\|\tilde{v}_1 - \tilde{v}_2\|_{L_2(B_R)} \le \underset{\substack{\downarrow R\to\infty \\ \infty}}{C_R}\|w_1 - w_0\|_{L_2(B_R)}$

$\sup_{t<T}\|\tilde{v}_1 - \tilde{v}_0\|_{L_2(B_R)} \le C_{\mu,R}T^{1/2}\sup_{t<T}\|v_1 - v_0\|_{L_2(B_R)}$

$w^R \in L_\infty(0,T; L_2(B_R)) \cap L_2(0,T; H_0^1(B_R))$ - nie zależy od R

Kłopot z przejściem z $R \to \infty$ – mamy mało informacji o v.

Dla ustalonego R

$$\begin{cases} w_t^R + v^R \nabla w^R - \mu \Delta w^R = 0 \\ w^R = 0 \quad \text{na} \quad \partial\Omega \end{cases}$$

$w_0^R |_{t=0} = w_0 \eta_R$ $\qquad$ $\inf w_0 \eta_R \leq w^R \leq \sup w_0 \eta_R$

$w_0 \in L_1 \cap L_\infty$

w^R – dobra funkcja testująca

$(w^R - k)_+$ – też będzie dobra

$$\frac{1}{2}\frac{d}{dt}\int_{B_R} (w^R - k)_+^2 + \mu \int |\nabla (w^R - k)_+|^2 dx \geq 0$$

$(w_R - k)|_{t=0} = 0$, $(w^R - k)_+ \equiv 0$

w_+^σ , $\sigma > 0$, $\sigma \to 0$ – problem, bo

(wszędzie tutaj $t = +$)

$$\nabla w_t^\sigma = \sigma \underbrace{w_t^{\sigma - 1}}_{!} \nabla w_t$$

$\nabla (w_t + L)^\sigma = \sigma (w_t + L)^{\sigma - 1} \nabla w_t$ – z tym też jest problem (ślad nie jest 0)

$(w_t + L)^\sigma - L^\sigma$ – tym już możemy testować

$$\int w_t ((w_+ + L)^\sigma - L^\sigma) = \int w_{+t} ((w_+ + L)^\sigma - L^\sigma)$$

$$B_R = \{ w > 0 \} \cup \{ w \leq 0 \}$$

$\frac{d}{dt} \int \phi_L (w_+)$, $L \to 0^+$

$$\frac{d}{dt} \int \frac{1}{1+\sigma} w_+^{1+\sigma} dx$$

Drugi człon po przecałkowaniu daje 0

$$-\int \mu \Delta w_+ [(w_+ + L)^\sigma - L^\sigma] dx = \mu \int \nabla w_+ \nabla w_+ [\sigma (w_+ + L)^{\sigma - 1}] \xrightarrow{L \to 0}$$

$$\to \mu \int |\nabla w_+|^2 \sigma w_+^{\sigma - 1}$$

$$\frac{1}{1+\sigma}\frac{d}{dt}\int_{B_R} w_+^{1+\sigma}\,dx + \mu\sigma\int_{B_R}|\nabla w_+|^2 w_+^{\sigma-1}\,dx \le 0$$

$$\downarrow \sigma\to 0$$

$$\frac{d}{dt}\int_{B_R} w_+\,dx + \mu\sigma\int_{B_R}|\nabla w_+|^2 w_+^{-1}\,dx \le 0$$

$$\int_{B_R} w_+\,dx \le \int_{B_R} w_{0+}\,dx$$

$$\|w\|_{L_\infty} \le \|w_0\|_{L_1\cap L_\infty}$$

w_R, v_R – loc

$w^R \rightharpoonup^* w$ w $L_\infty(0,T;L_\infty(\mathbb{R}^2))$

$w^R \rightharpoonup w$ w $L_2(0,T;\dot H^1(\mathbb{R}^2))$

$v_R = K_{BS} * Ew^R$

$v_R \in L_\infty(0,T; L_{2+\sigma}\cap W^1_{\infty-\sigma}(\mathbb{R}^2))$

$w^R \to w$ L_1

$v^R\nabla w^R$, żeby zbiegał potrzebujemy lepszej zwartości (po czasie)

Musimy pokazać $w_t^R \in \ldots$

Ta metoda nie działa, ale pokazujemy, że dla zadanego $v\in L_\infty(0,T;L_{2+\sigma}\cap W^1_{\infty-\sigma})$

$w_t + v\nabla w - \mu\Delta w = 0$

$w\in L_\infty(0,T;L_\infty(\mathbb{R}^2))$, $w\in L_\infty(0,T;L_2)\cap L_2(0,T;H^1(\mathbb{R}^2))$

$w \overset{?}{\in} L_\infty(0,T, L_{1+\sigma}(\mathbb{R}^2))$

Chcemy więcej!

$w|_{t=0} = w_0 \ge 0$

$$Ew_R = \begin{cases} w_R & x\in B_R \\ 0 & \text{w pp} \end{cases}$$ na ∂B_R jest źle

$w_0\eta_R \mapsto w^R$ B_R

Będziemy starać się badać $(w_{R+1}-w_R)_t + v\nabla(w_{R+1}-w_R) - \mu\Delta(w_{R+1}-w_R) \le 0$

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$$\begin{cases} \omega_t + v\nabla\omega - \mu\Delta\omega = 0 & \omega \quad \mathcal{D}'(\mathbb{R}^2) \\ \omega|_{t=0} = \omega_0 & , \quad \omega_0 \geq 0 \end{cases}$$

testowanie $(\omega-k)_\pm$ daje L_∞

Kłopot jest z L_1

$\omega \in L_\infty(0,T;L_1(\mathbb{R}^2))$

Interesuje nas $\|\omega_{R+1} - \omega_R\|_{L_1}$ $\quad \omega = \lim_{R\to\infty} \omega^R$

$$\omega_R = \begin{cases} \omega_R & \text{na } B_R \\ 0 & \text{poza} \end{cases}$$

$\omega = \sum(\omega_{R+1} - \omega_R)$ zbieżny w L_1

Pytamy się jak można oszacować $\omega_{R+1} - \omega_R$.

$$\int_\Omega (\omega_{R+1} - \omega_R)_t \varphi\, dx - \int_\Omega v\nabla(\omega_{R+1} - \omega_R)\varphi\, dx - \mu\int_\Omega (\omega_{R+1} - \omega_R)\Delta\varphi\, dx = ?$$

$$\int_{\Omega = B_{R+1} = B_R \cup (B_{R+1}\setminus B_R)} \dots = \int_{B_R} + \int_{B_{R+1}\setminus B_R}$$

$\omega_{R+1}, \omega_R \in L_2(0,T;H^1(\Omega))$

$$\int_{B_R} v\nabla(\omega_{R+1} - \omega_R)\varphi\, dx = \int_{\partial B_R} v\cdot n(\omega_{R+1} - \omega_R)\varphi\, d\sigma$$

$$\int_{B_{R+1}\setminus B_R} \dots = \int_{\partial B_R^-} v\cdot n^-(\omega_{R+1} - \omega_R)\varphi\, d\sigma$$

po dodaniu $= 0$ (bo przeciwne orientacje)

$$-\int_{B_{R+1}} (\omega_{R+1} - \omega_R)\Delta\varphi\, dx = -\left[\int_{B_{R+1}\setminus B_R} + \int_{B_R}\right] = \int_{B_{R+1}} \nabla(\omega_{R+1} - \omega_R)\nabla\varphi\, dx$$

$$\int_{\partial B_R^-} (\omega_{R+1} - \omega_R)\frac{\partial\varphi}{\partial n^-}\, d\sigma + \int_{\partial B_R^+} (\omega_{R+1} - \omega_R)\frac{\partial\varphi}{\partial n^+}\, d\sigma = 0$$

$$\int_{B_{R+1}\setminus B_R} + \int_{B_R} \nabla(\omega_{R+1} - \omega_R)\nabla\varphi\, dx \quad - \int_{B_{R+1}} \Delta(\omega_{R+1} - \omega_R)\varphi\, dx$$

$$\underbrace{+ \int_{\partial B_R^-} \frac{\partial(\omega_{R+1} - \omega_R)}{\partial n^-}\varphi\, d\sigma + \int_{\partial B_R^+} \frac{\partial(\omega_{R+1} - \omega_R)}{\partial n^+}\varphi\, d\sigma}$$

$$= \int_{\partial B_R} \left(-\frac{\partial\omega_{R+1}}{\partial n} + \frac{\partial\omega_{R+1}}{\partial n} - \frac{\partial\omega_R}{\partial n}\right)\varphi\, d\sigma = -\int_{\partial B_R} \frac{\partial\omega_R}{\partial n}\varphi\, d\sigma \geq 0$$

$$\int_{B_{R+1}} (w_{R+1} - w_R)_t \varphi\, dx - \int_{B_{R+1}} v \nabla\varphi (w_{R+1} - w_R)\, dx - \mu \int (w_{R+1} - w_R) \Delta \varphi\, dx =$$

$$= \Big(\int_{B_{R+1} \setminus B_R} + \int_{B_R} \Big) \Big((w_{R+1} - w_R)_t + v \nabla (w_{R+1} - w_R) - \mu \Delta (w_{R+1} - w_R) \Big) \varphi \overset{0}{=}$$

$$+ \int_{\partial B_R} \frac{\partial w_R}{\partial n} \varphi\, d\sigma \leq 0$$

($\geq$ czyli tak ~~chybadzi~~ wychodzi ...)

coś się ze znakiem nie zgadza...

<u>Modelowy problem</u> (chodzi o to samo):

$$\begin{cases} u_t - \Delta u = 0 \\ u = 0 \quad w\ B \\ u|_{t=0} = u_0 \geq 0 \end{cases}, \quad \varphi \in C_0^\infty(\mathbb{R}^n)$$

$$\int_{\mathbb{R}^n} (u_t \varphi - \Delta u \varphi)\, dx = \int_B u_t \varphi\, dx - \int_B \Delta u \varphi + \int_{\partial B} \frac{\partial \varphi}{\partial n}\, d\sigma \leq 0, \text{ bo:}$$

$$\int_{\mathbb{R}^d} - u \Delta \varphi\, dx = - \int_B u \Delta \varphi\, dx = \int_B \nabla u \nabla \varphi\, dx = - \int_B \Delta u \varphi\, dx + \underbrace{\int_{\partial B} \frac{\partial u}{\partial n} \varphi\, d\sigma}_{\leq 0}$$

$$\Rightarrow \int (u_t \varphi - u \Delta \varphi)\, dx \leq 0$$

– – –

Nie udało nam się w sposób ładny ~~pochoo~~ pokazać, że granica jest w L^1.

O powyższym należy zapomnieć:

$w_{R+1} - w_R \geq 0$

$w_{R+1} \geq w_R$

więc istnieje granica + lemat Fatou (tw. Lebesgue'a) $\Rightarrow$ ogr.

<u>Chcemy teraz pokazać istnienie nieliniowego:</u>

$v_0 \rightsquigarrow \tilde{v}_0$

$v_1 \rightsquigarrow \tilde{v}_1$

$$w_t + v_i \nabla w - \mu \Delta w = 0$$
$$\tilde{v}_i = B_{SV} w$$

$$(w_1 - w_0)_t + v_0 \nabla (w_1 - w_0) - \mu \Delta (w_1 - w_0) = (v_1 - v_0) \nabla w_1$$

$$\sup_{t\le T}\|w_1-w_0\|_{L_1} \le \int_0^T\int_{\mathbb{R}^2}|v_1-v_0|\,|\nabla w|\,dx\,dt \le \|\nabla w_1\|_{L_2 L_2} T^{1/2}\sup_t\|v_1-v_0\|_{L_2(\mathbb{R}^2)}$$

$$\sup_{t<T}\|w_1-w_0\|^2_{L_2} + \mu\int_0^t\int_{\mathbb{R}^2}|\nabla(w_1-w_0)|^2\,dx\,dt \le \int_0^T\int|v_1-v_0|\,w_1\,\nabla(w_1-w_0)|\,dx\,dt$$

$$\le \frac{\mu}{2}\|\nabla(w_1-w_0)\|^2_{L_2} + C_\mu|w_1|_\infty T^{1/2}\|v_1-v_0\|^2_{L_2}$$

coś się nie zgadza (potrzebujemy lepszych szacowań)

$$w_t = -v\nabla w + \Delta w$$

$$\int_{\mathbb{R}^2} w_t\cdot\varphi\,dx = \int_{\mathbb{R}^2}(vw-\nabla w)\nabla\varphi\,dx$$

$$\cap$$

$L_2(0,T;\dot H^{-1}(\mathbb{R}^2))$, $w_t\in\dot H^{-1}(\mathbb{R}^2)$

$$\begin{pmatrix} H^1\subset\dot H^1 \\ (H^1)^*\supset(\dot H^1)^*\end{pmatrix}$$

$w_t\sim \operatorname{div} K$ $\cancel{+K_a}$, $K\in L_2$

$v\in L_\infty L_2(\mathbb{R}^2)$

$w\in L_\infty L_\infty$

$\nabla w\in L_2$

$\operatorname{rot} v_t = w_t$ $\quad K\in L_2$

$\Delta\varphi_t = w_t = \operatorname{div} K$ $\quad$ testujemy po φ_t

$$\int|\nabla\varphi_t|^2 = \int K\nabla\varphi_t \le \int|K|^2$$

$$\shortparallel$$

$$\int|v_c|^2 \le \|K\|_{L_2}$$

$$\|(w_1-w_0)_t\|_{L_2(0,T;\dot H^{-1}(\mathbb{R}^2))} \le T^{1/2}\sup_t\|v_1-v_0\|_{L_2}\|w_1\|_{L_\infty} + \mu\|\nabla(w_1-w_0)\|_{L_2L_2}$$

$$+\|(w_1-w_0)v_1\|_{L_2L_2}$$

$$\le T^{1/2}\|w_1-w_0\|_{L_2}|v_1|_{L_\infty}$$

z popz. szacowań

$$\sup_{t<T}\|w_1-w_0\|_{L_2} + \|\nabla(w_1-w_0)\|_{L_2L_2} + \|(w_0-w_1)_t\|_{L_2\dot H^{-1}} \le$$

$$\le CT^{1/2}\sup_{t<T}\|v_1-v_0\|_{L_2}$$

$$\|(w_0-w_1)_t\|_{L_2\dot H^{-1}} \ge \|(\tilde v_1-\tilde v_0)_t\|_{L_2L_2} \ge T^{-1/2}\sup_t\|\tilde v_1-\tilde v_0\|_{L_2}$$

$$\sup_t\|\tilde v_1-\tilde v_0\|_{L_2} \le CT^{1/2}\|(v_1-v_0)\|_{L_2}$$

czyli dostaliśmy kontrakcję.

$$v_t - \Delta v + \nabla p = f$$
$$\operatorname{div} v = 0$$
$$v|_{t=0} = v_0$$

Przykładamy operator $\mathbb{P} = Id - R_i R_j$ i nasze r-nie ma postać

$$\begin{cases} (\mathbb{P}v)_t - \nu\Delta(\mathbb{P}v) = \mathbb{P}f \\ (\mathbb{P}v)|_{t=0} = \mathbb{P}v_0 \end{cases}$$

<u>Dekompozycja Helmholza</u> (na $\mathbb{R}^n$)

$v \in H^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$\Delta\varphi = \operatorname{div} v$

$v = (v - \nabla\varphi) + \nabla\varphi$

$\operatorname{div}(v - \nabla\varphi) = 0$

$\|\nabla\varphi\|_{L_2} \leq \|v\|_{L_2}$

$\|\nabla\varphi\|_{H^s(\mathbb{R}^n)} \leq c\|v\|_{H^s(\mathbb{R}^n)}$

$$-|\xi|^2\hat{\varphi} = i\xi^j v_j$$

$$\hat{\varphi} = -\frac{i\xi^j v_j}{|\xi|^2}$$

$$\widehat{D\varphi}^{(k)} = i\xi^k\hat{\varphi} = \frac{\xi^k\xi^j\hat{v}_j}{|\xi|^2} = \hat{R}_k\hat{R}_j\hat{v}_j$$

$$\mathbb{P} = Id - \frac{\xi_k\xi_l}{|\xi|^2}$$

$$(\widehat{\mathbb{P}v})^k = v^k - \sum_{k=1}^{d}\frac{\xi_k\xi_l}{|\xi|^2}\hat{v}_l$$

$$((I-P)f, Pg)_{H^s} = 0$$

$\mathbb{P}\nabla\varphi = 0 \qquad P \in S^0$

$\|Pf\|_{H^s} \leq \|f\|_{H^s}$

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$$u_t - \nu\Delta u + \nabla p = f \qquad \mathbb{R}^d,\ d=3$$
$$\operatorname{div} u = 0$$
$$u|_{t=0} = u_0$$

$$\mathbb{P} = Id - R_i R_j$$

$$\mathbb{P} v^j = v^j - \sum_{k=1}^{3} R_j R_k v^k$$

Uwaga: wykorzystujemy $\Omega = \mathbb{R}^d$

$$v \in \underbrace{\mathbb{P}(H^s)}_{H^s_{div}} = \{ v \in H^s(\mathbb{R}^d, \mathbb{R}^d) : \operatorname{div} v = 0 \ \text{ w } \ \mathcal{D}'(\mathbb{R}^d)\}$$

$$(\mathbb{P}v)_t - \nu \mathbb{P} v = \mathbb{P} f$$

$$v \in H^s_{div} : \ Pv \in H^s_{div}$$

$[P, \Delta] = 0$ $\quad H^s(\mathbb{R}^d)$ tylko jak jesteśmy w całej p-ni!

Nie zachodzi $[P, \Delta] \neq 0$ w $H^s(\Omega)$.

$$v_t - \nu \Delta v = Pf$$
$$v|_{t=0} = v_0$$

$$v_0 \in H^s_{div}(\mathbb{R}^d), \quad v(t) \in H^{\sqcup}_{div}$$

$$v(t) = S(t) v_0 + \int_0^t S(t-s)\, Pf(s)\, ds$$
$$S(t) = e^{\nu \Delta t}$$
$$\widehat{S(t)u} = e^{-\nu |\xi|^2 t} u \qquad \operatorname{div} u = 0$$

Tw: $\quad u_t - \Delta u = 0, \quad u|_{t=0} = u_0$

$u(t) = S(t) u_0$

$u_0 \in \dot{H}^s(\mathbb{R}^d)$, to $u(t) \in C([0,T]; \dot{H}^s(\mathbb{R}^d))$ $\qquad \|f\|^2_{\dot{H}^s} = \int |\xi|^{2s} |\hat{f}|^2 d\xi$

$tu(t) \in L_\infty(0,T; \dot{H}^{s+2}(\mathbb{R}^d))$ $\qquad \|f\|^2_{H^s} = \int (1+|\xi|)^{2s} |\hat{f}|^2 d\xi$

D-d: $\hat{v}(t) = e^{-\nu|\xi|^2 t} \cdot v_0$

$$\int_{\mathbb{R}^d} \phi(\xi) |\hat{v}|^2 d\xi = \int_{\mathbb{R}^d} \phi(\xi) e^{-2\nu|\xi|^2 t} |v_0|^2 d\xi,$$

czyli cz. 1 zachodzi dla $\cdot$ i bez $\cdot$

$$t^2|\xi|^2\phi(\xi)|u(t)|^2 = t^2|\xi|^4\phi(\xi)e^{-2\nu|\xi|^2 t}|u_0|^2$$

$$t^2|\xi|^4 e^{-2\nu|\xi|^2 t}$$

$$u^2 e^{-2\nu u} \qquad \text{- ogr.}$$

czyli może być bez $\cdot$

$u \in C(0,T;H^s(\mathbb{R}^d))$ ew $\dot{H}^s$

<u>Pytanie:</u> Kiedy

$$\int_0^t S(t-s)f(s)ds \in L_\infty(0,T;H^s(\mathbb{R}^d)) \quad ?$$

$$s = L_2 = \dot{H}^0 = H^0.$$

$$\left\| \int_0^t e^{-\nu|\xi|^2(t-s)}\hat{f}(s)ds \right\|_{L_2} \le \int_0^t \|e^{-\nu|\xi|^2(t-s)}\hat{f}(s)\|_{L_2} ds$$

$$\|e^{-\nu|\xi|^2(t-s)}\hat{f}(s)\|_{L_2} \le \begin{cases} c\|f\|_{L_2} \\ \frac{c}{t-s}\|f\|_{\dot{H}^{-2}} \\ \qquad (H^{-2}) \end{cases}$$

zatem

$$\|e^{-\nu|\xi|^2(t-s)}\hat{f}(s)\|_{L_2} \le \frac{c}{(t-s)^\alpha}\|f(s)\|_{\dot{H}^{-2\alpha}} \quad (H^{-2\alpha})$$

$\hat{\Box}$ Pokazać powyższą nierówność bez interpolacji

$$\left\| \int_0^t S(t-s)f(s)ds \right\|_{L_\infty(0,T;L_2)} \le c\cdot\int_0^t \frac{1}{(t-s)^\alpha}\|f\|_{L_\infty(0,T;\dot{H}^{-2\alpha})}ds \le$$

$$\le C_\alpha T^{1-\alpha}\|f\|_{L_\infty(0,T,\dot{H}^{-2s})}ds$$

<u>Uwaga:</u> Bez kropki nie zachodzi ta nierówność

<u>Idziemy do Naviera - Stokesa</u>:

$$u = S(t)u_0 + \int_0^t S(t-s)P\operatorname{div}(u\otimes u)ds$$

$$u - \nu\Delta u = -Pu\nabla u$$

<u>Tw:</u> $u_0 \in \dot{H}^s(\mathbb{R}^3)$ to istnieje jedyne $u \in C(0,T;\dot{H}^s(\mathbb{R}^3))$ dla $T>0$.

Patrzymy się $S(t-s)\,\mathbb{P}\,\mathrm{div}: X^{d\times d} \to X^d$

Oczywiste jest, że $S(t-s)\,\mathbb{P}\,\mathrm{div}: \dot{H}^{-1} \to L_2$

$$\| S(t-s)\mathbb{P}\,\mathrm{div} f \|_{L_2} \leq C \|f\|_{\dot{H}^{-1}}$$

$$\| S(t-s)\mathbb{P}\,\mathrm{div} f \|_{L_2} \leq \frac{C}{t-s} \|f\|_{\dot{H}^{-1}} \qquad / \cdot |\xi|^s$$

$$\| S(t-s)\mathbb{P}\,\mathrm{div} f \|_{\dot{H}^s} \lesssim \|f\|_{\dot{H}^{s+1}}$$

$$\| S(t-s)\mathbb{P}\,\mathrm{div} \|_{\dot{H}^s} \lesssim \frac{1}{t-s} \|f\|_{\dot{H}^{s-1}}$$

$$\boxed{\| S(t-s)\mathbb{P}\,\mathrm{div} \|_{\dot{H}^s} \lesssim \frac{1}{(t-s)^\alpha} \|f\|_{\dot{H}^{s+1-2\alpha}}} \quad (\text{z interpolacji}) \quad \alpha \in (0,1)$$

<u>Następny problem</u>

$u \in \dot{H}^s(\mathbb{R}^3)$, $s > 0$ (okaże się, że $s < \frac{3}{2}$)

Pytamy się do jakiego α: $u \otimes u \in H^\alpha(\mathbb{R}^3)$

Jeżeli $u \in \dot{H}^s(\mathbb{R}^3)$, to $u \in L_{\frac{6}{3-2s}}(\mathbb{R}^3)$.

Niech $s \in (0,1)$:

$$L_2 \to L_2$$
$$\dot{H}^1 \to L_6$$
$$(L_2, \dot{H}^1)_{\theta,2} \to (L_2, L_6)_{\theta,2} = L_{p,2} \subset L_p$$
$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{6}$$

$u \otimes u \in L_{\frac{3}{3-2s}} \overset{?}{\in} \dot{H}^\alpha(\mathbb{R}^3)$ α jest ujemne

Kiedy $H^{-\alpha} \subset L_{\left(\frac{3}{3-2s}\right)^*} = L_{\frac{3}{2s}}$

$$L_{\frac{3}{3-2s}} \cdot H^{-\alpha} \in L_1$$

$$-\alpha = \frac{3}{2} - 2s$$

Przyp. $H^{|\alpha|} \subset L_m$ $\quad \frac{3}{|\alpha|} \left|\frac{1}{2} - \frac{1}{m}\right| = 1$

$\dot{H}^{3/2-2s}(\mathbb{R}^3) \cap L_m(\mathbb{R}^3)$ jest prz. Banacha, gdy mamy

$H^s(\mathbb{R}^n) \subset L_\infty$ dla $s > \frac{n}{2}$

w $\mathbb{R}^3$ $s = \frac{3}{2}$

Wiemy, że $0 < s < \frac{3}{2}$ i chcemy:

$$s+1-2\alpha = -\frac{3}{2} + 2s \qquad [\text{patrz ramka}]$$

$$s = \frac{5}{2} - 2\alpha$$

$$\Rightarrow \quad \alpha \in \left(\tfrac{1}{2}, 1\right)$$

Jeżeli $u \in H^s(\mathbb{R}^3)$ $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$ $s = \frac{5}{2} - 2\alpha$, $\alpha \in \left(\frac{1}{2}, 1\right)$, to

$$\| S(t-s) P \operatorname{div} u \otimes u(s) \|_{H^s} \leq \frac{C}{(t-s)^\alpha} \| u(s) \|^2_{H^s}$$

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$$u_t - \Delta u = 0$$
$$u|_{t=0} = u_0$$

$$u(t) = S(t) u_0 = e^{\Delta t} u_0$$

$$\widehat{S(t) u_0} = e^{-\nu |\xi|^2 t} \hat{u}_0$$

$$u \in C([0,T); H^{-s}(\mathbb{R}^3))$$

$$u(t) = S(t) u_0 + \int_0^t S(t-s) \mathbb{P} \operatorname{div} (u \otimes u) ds$$

$$\underbrace{\| u \|_{L_\infty(0,T; \dot{H}^s)}}_{= X(T)} \leq \| u_0 \|_{\dot{H}^s(\mathbb{R}^3)} + \int_0^t \frac{c}{(t-s)^\alpha} \| u \|^2_{L_\infty(0,T; \dot{H}^s)} ds$$

$$\leq \| u_0 \|_{\dot{H}^s(\mathbb{R}^3)} + C_\alpha T^{1-\alpha} \| u \|^2_{L_\infty(0,T; \dot{H}^s)}$$

$$X(T) \leq X_0 + C_\alpha T^{1-\alpha} X^2(T)$$

$$0 \leq X_0 - X + C T^{1-\alpha} X^2$$

Jeśli T jest tak małe by

$$C_\alpha T^{1-\alpha} 4 \| u_0 \|_{\dot{H}^s} < 1, \quad \text{to} \quad X(T) \leq 2 \| u_0 \|_{\dot{H}^s}$$

$$X(T) \leq X_0 + \underbrace{C_\alpha T^{1-\alpha} 4 X_0}_{< 1} \cdot X_0 < 2 X_0$$

Jeżeli będziemy rozważać:

$$u_t - \nu \Delta u = - P \operatorname{div} \hat{u} \otimes \hat{u}$$
$$u|_{t=0} = u_0$$

i zdefiniujemy przestrzeń $\Xi(T) - v_0$

$$\Xi(T) = \{ v \in C([0,T); \dot{H}^s(\mathbb{R}^3)) : v|_{t=0} = v_0, \; \| v \|_{L_\infty(0,T, H^s)} \leq 2 \| v_0 \| \},$$ to

$$\|u\|_{L^\infty(0,T,\dot H^s)} \leq \|u_0\| + C_\alpha T^{1-\alpha} \|\tilde u\|^2_{L_\infty T \dot H^s} \leq 2\|u_0\|$$

$K: \Xi(T) \longrightarrow \Xi(T)$ chcemy pokazać kontrakcję

$K(\tilde u_0) = u_0$ $\qquad \delta u = u_1 - u_0$

$K(\hat u_1) = u_1$ $\qquad \delta\hat u = \tilde u_1 - \tilde u_0$

$$\delta u_t - \nu\Delta \delta u_t = -P\mathrm{div}(\hat u_0 \otimes \delta\tilde u) + (\hat u_1 \otimes \delta\hat u)$$

$$\|\delta u\|_X \leq C T^{1-\alpha} (\|\tilde u_0\| \|\delta \tilde u\| + \|\tilde u_1\| \|\delta\tilde u\|$$

$$\leq C T^{1-\alpha} (\|\tilde u_0\|_X + \|\tilde u_1\|_X) \|\delta\hat u\|_X$$

$$(C_\alpha T^{1-\alpha} 4\|u_0\|) \|\delta\hat u\|_X$$

$$< L\|\delta\hat u\| \quad , \quad L < 1$$

Głównym kłopotem istnienia globalnego w czasie jest całka

$$\int_0^t \underbrace{S(t-s) P\nabla(u\otimes u)}\, ds$$

jest całkowalne ?

$$\int_0^{t/2} S(t-s) P\mathrm{div}(u\otimes u)\, ds + \int_{t/2}^t S(t-s) P\mathrm{div}(u\otimes u)\, ds$$

$\downarrow \qquad\qquad\qquad \downarrow$

$\|S(\omega)f\|_{\dot H^{s+2}} \lesssim \frac{1}{\omega}\|f\|_{\dot H^s}$ $\qquad$ Można otarać się pokazać

$\|S(\omega)f\|_{\dot H^{s+4}} \leq \frac{1}{\omega^2}\|f\|_{\dot H^s}$ $\qquad$ $\|u(t)\|_{\dot H^s} \simeq \frac{1}{t^\beta}$, $\beta > \frac{1}{2}$

$$\|S(\omega)f\|_{\dot H^{s+2+2\varepsilon}} \leq \frac{C_\varepsilon}{\omega^{1+\varepsilon}}\|f\|_{\dot H^s}$$

Próbujemy rozbić $\int_0^{t-1} + \int_{t-1}^t$

$$\|S(t-s) P\mathrm{div} f\|_{\dot H^s} \leq \frac{1}{(t-s)^2}\|f\|_{\dot H^{s-3}}$$

$$\|S(t-s) P\mathrm{div} f\|_{\dot H^s} \leq \frac{1}{(t-s)^{1+\beta}}\|f\|_{\dot H^{s-1-2\beta}}$$

$u \in \dot H^s$, to $u\otimes u \in \dot H^{-(\frac{3}{2}-2s)}$, $-(\frac{3}{2}-2s) = s-1-2\beta$

$s = \frac{1}{2} - 2\beta$

leżymy i kwiczymy...

Gdyby zapomnieć o kropkach, to powinna wyjść nierówność

$$-\left(\tfrac{3}{2} - 2s\right) \geq s - 1 - 2\beta$$

$$s \geq \tfrac{1}{2} - 2\beta$$

$$\| S(t) w \|_{L_p} \leq \| w \|_{L_p}$$

$$\| S(t) w \|_{L_p} \leq C t^{\sqcup} \| w \|_{L_q} \qquad p \geq q \text{ , bo}$$

$$u(t) = \int \frac{1}{t^{n/2}} e^{-\frac{|x-y|^2}{t}} u(y)\, dy \qquad \text{(z dokł. do stałych)}$$

$$\| u \|_{L_p} \leq \left\| \frac{1}{t^{n/2}} e^{-\frac{x^2}{t}} \right\|_{L_r} \| u \|_{L_q} \qquad 1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{q}$$

$$\frac{1}{r} = 1 + \frac{1}{p} - \frac{1}{q}$$

$$\| \Delta S(t) \|_{L_q} \leq \frac{1}{t} \| w \|_{L_q} \qquad ?$$

$$\left(\int_{\mathbb{R}^n} \left| \frac{1}{t^{n/2}} e^{-\frac{x^2}{t}} \right|^r dx \right)^{1/r} = t^{-n/2} \left(\int (\sqrt{t})^n e^{-r w^2} dw \right)^{1/r}$$

$$= t^{-n/2 + \frac{n}{2r}}$$

$$= t^{-\frac{n}{2}\left(1 - \frac{1}{r}\right)}$$

$$= t^{-\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)}$$

$H^s(\mathbb{R}^3)$

$$u(t) = S(t) u_0 + \int_0^t S(t-s) P \operatorname{div}(u \otimes u)\, ds$$

$t < 1$. Gdzie musi należeć f tak żeby

$$\int_0^t S(t-s) P \operatorname{div} f\, ds \in L_2$$

z interpolacji dostajemy: $\| \nabla S(t) w \|_{L_q} \lesssim \frac{1}{t^{1/2}} \| w \|_{L_p}$

$$\| S(w) P \operatorname{div} f \|_{L_q} \lesssim t^{-1/2} \| f \|_{L_q}$$

$$\| S(w) g \|_{L_p} \lesssim t^{-\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} \| g \|_{L_q} \qquad \leftarrow \text{ tutaj } t = w$$

$$\| S(w) P \operatorname{div} f \|_{L_p} \leq t^{-\frac{1}{2} - \frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} \| f \|_q$$

$$\| S(\tfrac{w}{2}) S(\tfrac{w}{2}) P \operatorname{div} f \|_{L_p} \leq t^{-\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} \| S(\tfrac{w}{2}) P \operatorname{div} f \|_{L_q} \leq t^{-\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right) - \frac{1}{2}} \| f \|_{L_q}$$

<u>p=2</u> $\quad t^{-\frac{1}{2} - \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2}\right)}$ – ma być całkowalne w 0.

czy q może być blisko 2?

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Globalne w czasie rozwiązanie:

$S(t) = e^{\nu \Delta t}$ dla $\nu = 1$

$\| S(t) u \|_p \le \|u\|_p$

$\| \Delta S(t) u \|_p \lesssim t^{-1} \|u\|_p$

$S(t) = \Gamma *$

$\| \Delta \underbrace{\Gamma(\cdot, t) * u}_{Tu} \|_p \lesssim t^{-1} \|u\|_p$

$\widehat{Tu} = \underbrace{-|\xi|^2 e^{-|\xi|^2 t}}_{\text{nieogr.}} \hat{u} = -t^{-1} t |\xi|^2 e^{-|\xi|^2 t} \hat{u}$

tw. Marcinkiewicza o mnożnikach

$\| S(t) u \|_p \lesssim t^{-\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} \|u\|_q$

$S(t) u = \Gamma(\cdot, t) * u$

$\| \Gamma(\cdot, t) * u \|_p \le \| \Gamma(\cdot, t)\|_r \|u\|_q$

$1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{q}$

$\| \Gamma(\cdot, t)\|_r \simeq t^{-\frac{n}{2}} \left[\int_{\mathbb{R}^n} e^{-\frac{x^2}{t} r} dx \right]^{1/r}$

$\simeq t^{-\frac{n}{2}} \left[\int_{\mathbb{R}^n} t^{n/2} e^{-w^2} dw \right]^{1/r} \simeq c\, t^{-n/2\left(1 - \frac{1}{r}\right)} \simeq t^{-n/2\left(\frac{1}{q} - \frac{1}{p}\right)}$

$\Rightarrow \quad \| \nabla S(t) u \|_p \lesssim t^{-1/2} \|u\|_p$

Idziemy do NS. Bierzemy $u_0 \in H^s(\mathbb{R}^3)$, $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$

$u(t) = S(t) u_0 + \int_0^t S(t-s) \mathbb{P} \operatorname{div} (u \otimes u)\, ds$

Chcemy pokazać

$u \in C[0, T; H^s(\mathbb{R}^3))$, $T = \infty$, $\|u_0\|_{H^s}$ - małe

Zajmujemy się operatorem

$\underset{\nabla S(w)}{\overset{S(w) \mathbb{P} \operatorname{div}}{\shortparallel}}$, który jest tożsamy (bo $\Omega = \mathbb{R}^n$) z

$\| S(w) \mathbb{P} \operatorname{div} f \|_p \lesssim t^{-\frac{1}{2}} \|f\|_p$

$S(w) f = S\left(\frac{w}{2}\right) S\left(\frac{w}{2}\right) f$.

$$\| S(\omega)\mathbb{P}\,\mathrm{div}\, f\|_p \le \omega^{-1/2}\| S(\tfrac{\omega}{2}) f\|_p \le \omega^{-1/2-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|f\|_q$$

$$\| S(\omega)\mathbb{P}\,\mathrm{div}\, |\check{\xi}| f\|_p \lesssim \omega^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\| |\check{\xi}|^s f\|_p$$

$$\| \underset{I}{S(t)u}\|_p \le \|u\|_p\,, \quad \| \underset{|\xi|^2}{\Delta S(t)u}\|_p \le t^{-1}\|u\|_p$$

$$\| |\check{\xi}|^s S(t)u\|_p \le t^{-\frac{s}{2}}\|u\|_p$$

$$\| S(\omega)\mathbb{P}\,\mathrm{div}\, |\check{\xi}| f\|_p \lesssim \omega^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\| |\check{\xi}|^s f\|_q$$

$$\| S(\omega)\mathbb{P}\,\mathrm{div}\, f\|_{\dot{H}^s} \le \omega^{-\frac{1}{2}-\frac{s}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_q$$

$$\int_0^t = \int_0^{t-1} + \int_{t-1}^t \,, \quad t>1$$

$$\| S(t-s)\mathbb{P}\,\mathrm{div}\, u\otimes u\|_{L_2} \le (t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}\|u\otimes u\|_{L_q}$$

$$\|u\|_{H^s} = \|u\|_{L_2} + \|u\|_{\dot{H}^s}\,, \quad \text{bo} \quad (1+|\xi|^2)^{s/2} \underset{\ge}{\le} C(1+|\xi|^{2s})\,, \quad s\ge 0$$

$$\int_0^{t-1}\|\cdot\| \le C\|u\|^2_{L_\infty(0,T)} \quad ; H^s \subset \mathbb{R}$$

$$-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{2}\right) < -1\,, \quad \frac{1}{3} < \left(\frac{1}{q}-\frac{1}{2}\right)\,, \quad \frac{1}{q} > \frac{5}{6}\,, \quad q<\frac{6}{5} \qquad q = 1+\sigma$$

$$\|u\otimes u\|_{L_1} \le \|u\|^2_{L_2}$$

$$\int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})}\|u\otimes u\|_q$$

$$-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{2}\right) > -1\,, \quad q>\frac{6}{5}$$

$\#$ $u\otimes u \in L_q$ dla $q > \frac{6}{5}$?

$$u \in H^s \subset H^{1/2}\,, \quad s > \frac{1}{2}$$

<u>INTERPOLACJA</u>

$$H^{1/2} \underset{\mathbb{R}^3}{\subset} L_3 \cap L_2$$

$$L_2 \hookrightarrow L_2$$
$$H^1 \hookrightarrow L_6$$

$$H^{1/2} = (L_2, H^1)_{\frac{1}{2},1}$$

$$(L_2, L_6)_{\frac{1}{2},2} = L_{3,2} \subset L_3$$

$$\frac{1}{3} = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{6}$$

$$\Rightarrow \quad u\otimes u \in L_{\frac{3}{2}}$$
$$\frac{3}{2} > \frac{6}{5}$$

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$$\begin{cases} u_t - \Delta u = -\mathbb{P}\mathrm{div}(u\otimes u) \\ u|_{t=0} = u_0 \end{cases} \qquad u \in C([0,T]; H^s(\mathbb{R}^3)), \qquad s \in \left(\tfrac{1}{2}, \tfrac{3}{2}\right)$$

$\|u_0\|_{H^s(\mathbb{R}^3)} <<$ małe

Zadajemy pytanie: Czy $\|u(\cdot,t)\|_q \to 0$, $t \to \infty$,

czy szybko zbiega ?

$$\frac{1}{2}\frac{d}{dt}\int u^2 dx + \nu \int |\nabla u|^2 dx = 0$$

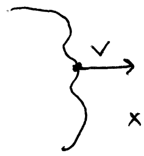
$$\frac{d}{dt}\int u^2 dx + \nu c \int u^2 dx \leq 0$$

$$\|u\|_{L_2(\Omega)}(t) \simeq e^{-\frac{\nu c}{2}t}\|u_0\|_{L_2}$$

$u \in C^\infty(\Omega \times (0,\infty))$

$|u|_{H^\infty} < \infty$

$\|u\|_2 \to e^{-t}$



$x(y,t) = y + \int_0^t v\vec{n}\, ds$

$\|v\| \neq 0$

$$\|u\|_{L_\infty} \leq \|u\|_{H^{\frac{n}{2}+\varepsilon}} \leq \|u\|^{\theta}_{\dot{H}^{2n+100}} \|u\|^{1-\theta}_{L_2} - \|u\|_{L_2}$$

$e^{-(1-\theta)t}$, $\quad e^{-t}$

$\theta \in (0,1)$

$$\Rightarrow \int_0^\infty |u|_\infty ds < \infty$$

Rozpatrujemy

$$\sup\, (t+1)^\alpha \|u\|_{H^r(\mathbb{R}^3)} \qquad H^{3/2}(\mathbb{R}^3) \not\hookrightarrow L_\infty$$

$$\|S(t)u\|_p \lesssim t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u\|_q, \qquad q=2,\ p=\infty$$

$$\|S(t)u\|_\infty \leq t^{-\frac{n}{4}}\|u\|_2$$

$$\int_0^{t/2} + \int_{t/2}^t$$

$$\int_0^{t/2} S(t-s)\mathbb{P}\,\mathrm{div}(u\otimes u)\,ds = \int_0^{t/2} S\left(\tfrac{t-s}{2}\right)S\left(\tfrac{t-s}{2}\right)\mathbb{P}\,\mathrm{div}(u\otimes u)\,ds$$

$u \sim (1+t)^\alpha \|u\|_{\dot{H}^r}$ $\qquad \dot{H}^r \subset L_m \qquad \frac{m}{2} = q, \quad m = \frac{6}{3-2r}, \quad q = \frac{3}{3-2r}$

korzystamy z nierówności $\quad \|S(\omega)\mathbb{P}\,\mathrm{div} f\|_{\dot{H}^s} \leq \omega^{-\frac{1}{2}-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\|f\|_q$

$$\int_0^{t/2}\|\ \ \|_{\dot{H}^r} \leq t^{-\frac{1}{2}-\frac{r}{2}-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\int_0^{t/2}\|u\otimes u\|_q\, ds$$

$$\int_0^{t-1} (t-s)^{-\frac{1}{2} - \frac{s}{2} - \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2}\right)} \|u \otimes u\|_{L_q} \le \|u\|^2_{L_\infty L_2}$$

$$-\frac{1}{2} - \frac{s}{2} - \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2}\right) < -1$$

$$1 = \frac{1}{q} > \frac{5}{6} - \frac{s}{3}$$

$$\int_{t-1}^{t} (t-s)^{-\frac{1}{2} - \frac{s}{2} - \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2}\right)} \|u \otimes u\|_q$$

$$-\frac{1}{2} - \frac{s}{2} - \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2}\right) > -1$$

$$\frac{1}{q} < \frac{5}{6} - \frac{s}{3}$$

$H^s \subset L_\omega$?

$H^s \subset L_{\frac{6}{3-2s}}$ $\qquad \frac{1}{2} < s < \frac{3}{2}$

$$q = \frac{3}{3-2s}$$

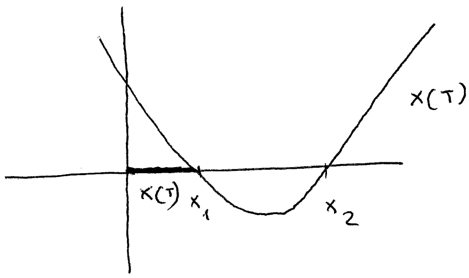
$u \otimes u \in L_{\frac{3}{3-2s}}$ $\qquad \frac{3-s}{3} < \frac{5}{6} \Rightarrow s > \frac{1}{2}$

$$\|u\|_{L_\infty(0,T,H_s)} \le \|u_0\|_{H_s} + C\|u\|^2_{L_\infty(0,T;H^s)}$$

$$x(T) \le x(0) + Cx^2(T)$$

$$0 \le x(0) - x(T) + Cx^2(T)$$

$x_{1,2}(T) = \dfrac{1 \pm \sqrt{1 - 4x_0 C}}{2C}$ o ile $4x_0 C < 1$



$u \in C(0,T,H^s)$. Co to znaczy?

<u>Tw.</u> $u_0 \in H^s(\mathbb{R}^3)$, $\|u_0\|_{H^s} << 1$, to istnieje jedyne, łagodne (mild) rozwiązanie

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\mathbb{P}\operatorname{div}(u \otimes u)\,dt$$

$$\int_0^{t/2} \|u\otimes u\| \, ds \leq \int_0^{t/2} (1+s)^{-2\alpha} (1+s)^{2\alpha} \|u\|^2_{\dot H^r} \, ds \quad , \quad \alpha > \frac{1}{2}$$

$$-\frac{1}{2} - \frac{r}{2} - \frac{3}{2}\left(\frac{3-2r}{3} - \frac{1}{2} \right) = -\frac{5}{4} + \frac{r}{2}$$

$$-\frac{5}{4} + \frac{r}{2} < -\frac{1}{2} \quad \Rightarrow \quad r < \frac{3}{2}$$

$$\int_{t/2}^{t} \| S(t-s) \mathbb{P}\mathrm{div}\,(u\otimes u) \|_{\dot H^r} \, ds$$

$$\| S(t-s) \mathbb{P}\mathrm{div}\,(u\otimes u) \|_{\dot H^r}$$

$$\int_{t/2}^{t} \frac{1}{(t-s)^{\frac{1}{2}+\varepsilon}} \frac{1}{s^{\frac{1}{2}+\varepsilon}} \sim t^{-\frac{1}{2}-\delta}$$

$$\downarrow$$

$$t^{-\frac{1}{2}-\varepsilon} \quad t^{+\frac{1}{2}-\varepsilon} \sim t^{-2\varepsilon} \quad , \quad \varepsilon > \frac{1}{4}$$

$$\| S(t-s) \mathbb{P}\mathrm{div}\,(u\otimes u) \|_{\dot H^r} \lesssim (t-s)^{-\frac{1}{2} - \frac{r}{2} - \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2}\right)} \|u\otimes u\|_q$$

kiedy : $-\frac{1}{2} - \frac{r}{2} - \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2} \right) < -\frac{3}{4}$?

$$\frac{1}{q} = \frac{1-\theta}{1} + \frac{\theta(3-2r)}{3}$$

$$1 - \frac{1}{q} = \frac{\theta \, 2r}{3} \qquad \theta = \frac{3}{2r}\left(1 - \frac{1}{q}\right)$$

$$\|u\otimes u\|_q \leq \|u\otimes u\|_{L_1} \|u\otimes u\|_{\frac{m}{2}} = \frac{3}{3-2r}$$

$$2\alpha \frac{3}{2r}\left(1 - \frac{1}{q}\right) > \frac{3}{4} \quad , \quad \alpha > \frac{1}{2}$$

$$\frac{r}{2} + \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2} \right) > \frac{1}{4} \qquad 1 - \frac{1}{q} > \frac{r}{2} \qquad r = \frac{1}{2}$$

$$q < \frac{4}{3}$$

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Asymptotyka rozwiązań w 3d:

Rozwiązania NS w 3D:

$$\begin{cases} u_t - \Delta u = -P\,\mathrm{div}\,(u\otimes u) \\ u|_{t=0} = u_0 \end{cases}$$

$u_0 \in H^s(B^3)$, $s \in (\frac{1}{2}, \frac{3}{2})$, $\|u_0\|_{H_s} << 1$

$\|u(\cdot,t)\|_{\dot H^s} \searrow 0$

$\sup_t t^\alpha \|u(\cdot,t)\|_{\dot H^s} = X$

$\sup_t \|u(\cdot,t)\|_{H^s} = M$

$u(t) = S(t)u_0 - \int_0^t S(t-s)\,P\,\mathrm{div}\, u\otimes u\, ds$

$t > 2$

$$\int_0^t = \int_0^1 + \int_1^{t/2} + \int_{t/2}^t$$

$\| S(t-s) P\,\mathrm{div}\,(u\otimes u)\|_{\dot H^r} \lesssim (t-s)^{-\frac{r}{2}-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})} \|u\otimes u\|_{L_q}^{-1}$, $1 \le q \le 2$

$s \in (1, \frac{t}{2})$

$\dot H^r \subset L_{\frac{6}{3-2r}}$

$\dot H^r \cdot \dot H^r \subset L_{\frac{3}{3-2r}}$ $\qquad -\frac{r}{2} - \frac{1}{2} - \frac{3}{2}\left(\frac{3-2r}{3} - \frac{1}{2}\right) < -\frac{1}{2}$

$\Rightarrow r < \frac{3}{2}$

$q = 1$ $\qquad t^{-\frac{r}{2}-\frac{1}{2}-\frac{3}{4}} \int_0^1 \|u\otimes u\|_{L_1} \le t^{-\alpha} M^2$

$s \in (\frac{t}{2}, t)$

$\|u\otimes u\|_{L_2} \le \|u\otimes u\|_{L_1}^{1-\Theta} \|u\otimes u\|^{\Theta}_{L_{\frac{3}{3-2r}}}$

$\frac{1}{2} = 1-\Theta + \Theta\frac{3-2r}{3}$, $\Theta = \frac{3}{4r}$

$\int_{t/2}^t \|S(t-s) P\,\mathrm{div}\, u\otimes u\|\, ds \lesssim \int_{t/2}^t (t-s)^{-\frac{r}{2}-\frac{1}{2}} s^{-\frac{3\alpha}{2r}} M^{2-2\Theta} X^{2\Theta}$

$\int_{t/2}^t (t-s)^{-\frac{r}{2}-\frac{1}{2}} s^{\frac{3\alpha}{2r}}\, ds \lesssim t^{-\frac{3}{2r}\alpha}\, t^{\frac{1}{2}-\frac{r}{2}}$, α

$-\frac{3}{2r}\alpha + \frac{1}{2} - \frac{r}{2} < -\alpha$ działa dla $r = 1$ $\qquad |\; X \le X_0 + cX^2 + cX^{2\Theta}M^{2-2\Theta} + cM^2$