Some regularity results to the generalized Emden–Fowler equation with irregular data

Agnieszka Kałamajska and Katarzyna Mazowiecka *

Faculty of Mathematics, Informatics, and Mechanics, University of Warsaw, ul. Banacha 2, 02–097 Warszawa, Poland

email addresses: (A.K.) kalamajs@mimuw.edu.pl, (K.M.) k.mazowiecka@mimuw.edu.pl

Abstract

We deal with the generalized Emden–Fowler equation $f''(x) + g(x)f^{-\theta}(x) = 0$, where $\theta \in \mathbb{R}$, $x \in (a,b)$, g belongs to $L^p((a,b))$. We obtain *a priori* estimates for the solutions, as well as information about their asymptotic behavior near boundary points. As a tool we derive new nonlinear variants of first and second order Poincaré inequalities, which are based on strongly nonlinear multiplicative inequalities obtained recently in [11].

Key words and phrases: Emden–Fowler equation, regularity, multiplicative inequalities, Poincaré inequalities, a priori estimates, nonlinear eigenvalue problems, singular differential equations.

MSC (2000): Primary 34B16, 46E35, 26D10, 34B15, 34B18, 34E10

1 Introduction

In this paper we deal with the generalized Emden–Fowler equation

$$f''(x) + g(x)f^{-\theta}(x) = 0, \text{ where } \theta \in \mathbb{R}, \ x \in (a,b), \tag{1.1}$$

which appears in many branches of mathematical physics.

For $\gamma := -\theta > 0$ the equation is known as the generalized Emden–Fowler equation with positive exponent and is of great interest in literature. It appears in the study

*The work of both authors is supported by the Polish Ministry of Science grant no. N N201 397837 (years 2009-2012).

of gas dynamics, fluid mechanics, relativistic mechanics, nuclear physics and in study of chemically reacting systems. In particular the equation $y''(t) = t^{\frac{1}{2}}y(t)^{\frac{3}{2}}$ found independently by Thomas [24] and Fermi [8] describes the electrical potential in an isolated neutral atom. For a detailed account of historical developments, particular applications of this equation we refer to the survey paper of Wong [27, Section 2].

For $\gamma := -\theta < 0$ the equation is called the generalized Emden–Fowler equation with negative exponent. It also arises in many branches of applied mathematics. For example it appears in fluid dynamics to investigate problems in non-Newtonian fluid flow [14] or to describe the flow over an impermeable plate [18]. Equations of this type are found in permeable catalysis [22], are used to describe glacial advance [26] or the transport of coal slurries down conveyor belts [7].

Let us mention the following two problems that arises in the study of Emden–Fowler equations:

A) the existence and uniqueness of solutions;

B) the regularity and qualitative properties of solutions, involving questions about their asymptotic behavior and *a priori* estimates.

We briefly discuss some selected results considering cases A) and B) separately.

A) THE EXISTENCE AND UNIQUENESS. In literature authors usually consider classical solutions to (1.1), i.e. solutions in the class $C([0,1]) \cap C^2((0,1))$, under various boundary conditions and assumption that $g \in C((a,b))$, $g > 0$. For example Nachman and Callegari [18] proved for $g(x) = x$ and $\theta = 1$ the existence, uniqueness and analyticity of positive solutions with vanishing Dirichlet boundary conditions. A necessary and sufficient condition on g for the existence of solutions to (1.1) with $\theta > 0$ was given by Taliaferro [23], who also proved that such solutions are unique. For a deeper discussion see [21] and for results regarding more general equations see [2, 9, 10, 20].
A necessary and sufficient condition, obtained in [23] for the existence of classical solutions in case when g is continuous and positive on (a,b), reads as: $\int_a^b t(1-t)g(t)dt < \infty$. It allows function g to be unbounded as $x \to a^+$ or $x \to b^-$. It seems natural to ask whether function g may blow-up or vanish in the interior points of the domain as well. To the best of our knowledge there are no results in this direction. On the other hand it is not difficult to find an equation of the form (1.1) and dealing with an irregular function g. For example the equation is satisfied (in the sense of Definition 6.1) with $g(x) = \alpha(\alpha-1)x^{\alpha-2+\alpha\theta}\chi_{x>0}$, $f(x) = x^\alpha\chi_{x>0}$, when $\alpha \in (0,1)$ and $\theta < -1 + \frac{1}{\alpha}\left(2 - \frac{1}{p}\right)$ (see Example 6.1 for details).

B) THE REGULARITY AND QUALITATIVE PROPERTIES OF SOLUTIONS. Some authors deal with the qualitative properties of solutions to (1.1). We only discuss results involving $g \in C((a,b))$ as we did not find any other ones. In this case solutions are classical, so their regularity is $C([a,b]) \cap C^2((a,b))$. The authors often ask about the

oscillatory properties of solutions (see e.g. [28]) or their asymptotic behavior ([19, 23]). Naito in [19] considered the case $(a,b) = (0,\infty)$ and obtained the conditions on admitted continuous functions g such that the solutions to (1.1) behave like nontrivial affine functions $c_1x + c_2$ as $x \to \infty$. The study of asymptotic behavior of solutions to (1.1) in case $\theta > 0$, g continuous and $(a,b) = (0,1)$ (in particular (a,b) is bounded), was provided in [23]. Some other related results can be found in [5, 25, 29].
It is natural to ask what can be said about the asymptotic behavior of the solutions in case when g is less regular than continuous. This question is also of our interest.

Our goal is to present an effective tool to study weak solutions to (1.1) in case when $g \in L^p((a,b))$, $p \geq 2$, $\theta \in \mathbb{R}$, f is nonnegative and not necessarily strictly positive. We achieve it in several approaches, which are described below.

- When dealing with (1.1) we have to *introduce a new definition of solutions*. The difficulty there is that when g is irregular the solution may be irregular as well. By our general assumptions f is continuous but it may not belong to Sobolev space $W^{1,1}_{loc}((a,b))$ (see Theorem 6.1). In particular f'' is defined in distributional sense on whole interval (a,b), but when θ is positive function $gf^{-\theta}$ is defined only when f is nonzero. This is why we have to introduce a new definition of solutions when $\theta > 0$ and f may admit zeroes inside (a,b) (Definition 6.1). Under our assumptions, in general, the strong maximum principle does not apply to such solutions (cf. Example 6.1) and indeed the solutions may admit zeroes in (a,b). Moreover, our class of admitted functions is essentially larger than the class in [11], where the authors assumed that $f \in W^{2,1}_{loc}((a,b))$.

- As a main result *we obtain the following a priori estimates for solutions* (cf. Theorem 6.1):

$$\int_{\{f>0\}} (f(x))^p (f(x))^{\theta p} dx \leq C \int_{\{f>0\}} |g(x)|^p dx, \tag{1.2}$$

$$\int_{\{f>0\}} |f'(x)|^p (f(x))^{\theta p} dx \leq C \int_{\{f>0\}} |g(x)|^p dx. \tag{1.3}$$

- The estimates are based on *first and second order strongly nonlinear Poincaré inequalities* derived here which seem to be new (Theorems 4.1 and 5.1):

$$\begin{aligned} \int_{\{f>0\}} (f(x))^p (f(x))^{\theta p} dx \leq& C \int_{\{f>0\}} |f'(x)|^p \, (f(x))^{\theta p} dx \text{ and} \\ \int_{\{f>0\}} (f(x))^p (f(x))^{\theta p} dx \leq& C \int_{\{f>0\}} |f''(x)|^p (f(x))^{\theta p} dx, \\ \int_{\{f>0\}} |f'(x)|^p (f(x))^{\theta p} dx \leq& C \int_{\{f>0\}} |f''(x)|^p (f(x))^{\theta p} dx. \end{aligned} \tag{1.4}$$

- The second order Poincaré type inequalities are obtained as a consequence of a modification of the following *strongly nonlinear multiplicative inequality* obtained recently in [11] (cf. Theorem 3.1):

$$\int_{\{f>0\}} |f(x)'|^p (f(x))^{\theta p} dx \le \left(\frac{p-1}{|1+\theta p|} \right)^{\frac{p}{2}} \int_{\{f>0\}} (\sqrt{|f(x) f''(x)|})^p (f(x))^{\theta p} dx. \tag{1.5}$$

- As an example of application of our regularity results (1.2) and (1.3) we deduce *asymptotic behavior of solutions* to (1.1) near the boundary point a, generalizing some of the earlier estimates from [11] (cf. Theorem 6.1). Contrary to the similar estimates contributing to the study of the asymptotic behavior proven in [11], we do not assume here that the solutions to (1.1) are strictly positive nor that $f \in W^{2,1}_{loc}((a,b))$. For example we deduce that when $f(0) = f(1) = 0$ and $\theta > -1$, we have

$$0 \le f(x) \le C x^{\frac{p-1}{p(1+\theta)}} \left(\int_a^b |g(x)|^p dx \right)^{\frac{1}{p(1+\theta)}}$$

  as $x \to 0$. In particular when $-1 < \theta < -\frac{1}{p}$ the convergence to zero is faster than linear. See Remark 7.5 for details. As we link our estimates with L^p norm of g, our results cannot be compared directly with [5, 23, 25, 29].

As a partial result we obtain that a solution to (1.1) is of the form $h(x)^{\frac{1}{1+\theta}}$, where $h \in W^{1,p}((a,b))$, see Theorem 6.1. This suggests that perhaps when dealing with singular ODEs it is natural to expect that solutions are rather compositions of Sobolev functions than Sobolev functions itself. This is also why we required different than usual definition of solutions. So far the problem of characterizing the set X satisfying the condition: $f \in X \Leftrightarrow T(f) \in W^{k,p}$, where T is the given mapping, is not well understood, see [3, 6] for the related results.

2 Preliminaries and notation

Notation. If $I \subseteq \mathbb{R}$ is an open subset, we use the standard notation: $C_0^\infty(\mathrm{I})$ to denote smooth compactly supported functions, $W^{m,p}(\mathrm{I})$ and $W^{m,p}_{loc}(\mathrm{I})$ to denote the spaces of global and local Sobolev functions defined on I, respectively. If $A \subseteq \mathbb{R}$ and f is defined on A we denote by $f\chi_A$ the extension of f by zero outside set A. We write $B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt$ to denote Euler's beta function and $\mathcal{D}'(I)$ to denote the space of distributions on I. When $1 < p < \infty$, by p' we denote Hölder conjugate to p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. We write $f_1 \sim f_2$ if there exists universal constants $C_1, C_2 > 0$ such that

$C_1 f_2 \le f_1 \le C_2 f_2$ on their domain. We set

$$\Phi_p(\lambda) = \begin{cases} |\lambda|^{p-2}\lambda & \text{for} \quad \lambda \neq 0 \\ 0 & \text{for} \quad \lambda = 0 \end{cases}$$

for $\lambda \in \mathbb{R},\ p > 1$. An easy verification shows that Φ_p is a continuous function.
We also define for $\theta \neq -1$:

$$T(\lambda) = \begin{cases} \frac{1}{\theta+1}\lambda^{\theta+1} & \text{when} \quad \lambda > 0 \\ 0 & \text{when} \quad \lambda \le 0 \end{cases} \text{ and } \tau(\lambda) = \begin{cases} \lambda^{\theta} & \text{when} \quad \lambda > 0 \\ 0 & \text{when} \quad \lambda \le 0. \end{cases} \tag{2.1}$$

It is clear that T is differentiable on $(0, \infty)$ and $T' = \tau$ on that set.

Poincaré inequalities.
We will be using the following Poincaré inequality. We submit the proof for readers convenience.

Proposition 2.1. *Let $-\infty < a < b < +\infty$, $1 < p < \infty$. Then inequality*

$$\int_a^b |f(x)|^p dx \le C_p(a,b) \int_a^b |f'(x)|^p dx \tag{2.2}$$

holds with best constant $C_p(a,b)$, where

i) $C_p(a,b) = (b-a)^p \left(\frac{p^{1/p'}(p')^{1/p}}{B(1/p,1/p')}\right)^p = \frac{\left(p(b-a)\sin\left(\frac{\pi}{p}\right)\right)^p}{(p-1)\pi^p}$ *for*
$f \in Z^{1,p}((a,b)) := \{w \in W^{1,p}((a,b)) : w(a) = 0 \text{ or } w(b) = 0\}$,

ii) $C_p(a,b) = 2^{-p}(b-a)^p \left(\frac{p^{1/p'}(p')^{1/p}}{B(1/p,1/p')}\right)^p = \frac{\left(p(b-a)\sin\left(\frac{\pi}{p}\right)\right)^p}{2^p(p-1)\pi^p}$ *for $f \in W_0^{1,p}((a,b))$.*

Proof. For part ii) see e.g. [4, page 180]. To prove part i) we first observe that it suffices to obtain the statement in case $a = 0, b = 1$. We will deal with the following objects:

$$W^{1,p}_{0,sym}((-1,1)) := \{w \in W_0^{1,p}((-1,1)) : w(x) = w(-x)\},$$
$$Z_r^{1,p}((0,1)) := \{w \in W^{1,p}((0,1) : w(1) = 0\},$$
$$J(u) := \frac{\int_{-1}^1 |u'(\tau)|^p d\tau}{\int_{-1}^1 |u(\tau)|^p d\tau}, \quad u \not\equiv 0,$$
$$I(u) := \frac{\int_0^1 |u'(\tau)|^p d\tau}{\int_0^1 |u(\tau)|^p d\tau}, \quad u \not\equiv 0.$$

We observe that when $u \in W_0^{1,p}((-1,1))$ is nonnegative then the function $v(x) = \left(\frac{u^p(x)+u^p(-x)}{2}\right)^{\frac{1}{p}}$ satisfies $\int_{-1}^1 v^p dx = \int_{-1}^1 u^p dx$ and $|v^{'}(x)|^p \le \frac{1}{2}\left(|u^{'}(x)|^p + |u^{'}(-x)|^p\right)$ (this argument is a simple modification of the argument taken from [13, proof of Lemma 3.3]), therefore $\int_{-1}^1 |v^{'}|^p dx \le \int_{-1}^1 |u^{'}|^p dx$ and

$$\mathcal{A} := \inf\{J(u) : u \in W_0^{1,p}((-1,1)), u \not\equiv 0\} = \inf\{J(u) : u \in W_{0,sym}^{1,p}((-1,1)), u \not\equiv 0\}.$$

Moreover, the mapping $u \mapsto \hat{u}$ where $\hat{u}$ is given by $\hat{u}(x) = \begin{cases} u(x) & \text{when } x \in [0,1], \\ u(-x) & \text{when } x \in [-1,0) \end{cases}$ is a surjection of $Z_r^{1,p}((0,1))$ onto $W_{0,sym}^{1,p}((-1,1))$ and we have $J(\hat{u}) = I(u)$. Hence

$$\mathcal{A} = \inf\{I(u) : u \in Z_r^{1,p}((0,1)), u \not\equiv 0\} = \inf\{I(u) : u \in Z^{1,p}((0,1)), u \not\equiv 0\}.$$

As already shown in ii) we have $\mathcal{A} = \left(\frac{p^{1/p'}(p')^{1/p}}{B(1/p,1/p')}\right)^{-p}$ and the statement follows. □

Remark 2.1. As is well known (see [17, Sections 1.1 and 1.4]), finiteness of the right-hand side in (2.2) and assumption $f \in W_{loc}^{1,1}((a,b))$ imply $f \in W^{1,p}((a,b)) \cap C([a,b])$.

Nonlinear Sobolev spaces.
We introduce the following possibly nonlinear Sobolev and Beppo–Levi type "spaces".

Definition 2.1. *Let $m \in \mathbb{N}$, $p \ge 1$, $-\infty \le a < b \le +\infty$, $\theta \in \mathbb{R}$.*

i)(nonlinear Sobolev spaces) *By $W^{m,p,\theta}((a,b))$ we will denote the subset of $W_{loc}^{m,1}((a,b))$ consisting of those functions, for which*

$$\sum_{k=0}^m \int_{(a,b)\cap\{x:f(x)\neq 0\}} |f^{(k)}(x)|^p |f(x)|^{p\theta} dx < \infty.$$

ii)(nonlinear Beppo–Levi spaces) *By $L^{m,p,\theta}((a,b))$ we will denote the subset of such functions in $W_{loc}^{m,1}((a,b))$, which satisfy the condition*

$$\int_{(a,b)\cap\{x:f(x)\neq 0\}} |f^{(m)}(x)|^p |f(x)|^{p\theta} dx < \infty.$$

We also define the local spaces $W_{loc}^{m,p,\theta}((a,b))$ and $L_{loc}^{m,p,\theta}((a,b))$ in the natural way, i.e. f belongs to the related local space $X_{loc}((a,b))$ if for any $[a^{'},b^{'}] \subseteq (a,b)$ we have $f \in X((a^{'},b^{'}))$. Analogously we define these spaces on an arbitrary open set in $\mathbb{R}$.

Our considerations will be restricted to the case $m \in \{1,2\}$ and to nonnegative functions.

Compositions of Sobolev functions.
We will be using the following well known fact (see e.g. [17]).

Lemma 2.1. *If $f : [-R,R] \to \mathbb{R}$ is absolutely continuous with values in the interval $[\alpha,\beta]$ and $L : [\alpha,\beta] \to \mathbb{R}$ is Lipschitz, then the function $(L \circ f)(x) := L(f(x))$ is absolutely continuous on $[-R,R]$.*

3 Strongly nonlinear multiplicative inequalities

Our consideration will be based on the following theorem which is a modification of a strongly nonlinear multiplicative inequalities obtained in [11, Propositions 4.2 and 4.3] where assumption $f \in W^{2,1}_{loc}((a,b))$ is weakened to $f \in W^{2,1}_{loc}(I_f)$.

Theorem 3.1. *Let $-\infty < a < b < \infty$, $p \geq 2$, $\theta \in \mathbb{R}$, $\theta \neq -\frac{1}{p}$, $\Phi_p(\cdot)$ be given by (2.1). Moreover, let function $f : (a,b) \to \mathbb{R}$ be nonnegative, continuous and such that*

a) $f \in W^{2,1}_{loc}(I_f)$ where $I_f = \{x \in (a,b) : f(x) > 0\}$. In particular function

$$\mathcal{A}f(x) := \frac{1}{\theta p + 1}\Phi_p(f'(x))(f(x))^{\theta p+1}\chi_{\{f>0\}}(x) \tag{3.1}$$

is well defined;

b) in case $\theta < -\frac{1}{p}$ let function f be either strictly positive or function $\mathcal{A}f$ be continuous on (a,b).

Then for every r, R such that $a < r < R < b$, we have

$$\int_{\{x\in(r,R):f(x)>0\}} |f'(x)|^p (f(x))^{\theta p} dx \leq \tag{3.2}$$
$$\left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \int_{\{x\in(r,R):f(x)>0\}} \left(\sqrt{|f(x)f''(x)|}\right)^p (f(x))^{\theta p} dx + \tilde{\Theta}(r,R),$$

where $\tilde{\Theta}(r,R) := \mathcal{A}f(R) - \mathcal{A}f(r)$.

Remark 3.1. Obviously, when the right-hand side in (3.2) is finite for every $a < r < R < b$, we necessarily have $f \in W^{2,p/2}_{loc}(I_f)$.

Remark 3.2. If $x \in (a,b)$ belongs to the boundary of I_f, i.e. $f(x) = 0$ then $f'(x)$ might not be defined according to our assumptions. In such a case
$\mathcal{A}f(x) = \frac{1}{\theta p+1}\Phi_p(f'(x))(f(x))^{\theta p+1}\chi_{\{f>0\}}(x) = 0$.

Proof. We start by recalling Lemma 4.1 of [11] with $h(\lambda) = \lambda^{\theta p}$.

Lemma 3.1 ([11])**.** *Let $-\infty \leq a < b \leq \infty$, $p \geq 2$, $\theta \in \mathbb{R}$, $\theta \neq -\frac{1}{p}$, $\eta > 0$. Then for every $f \in W^{2,1}_{loc}((a,b))$ such that $f \geq \eta$ and for every r, R such that $a < r < R < b$, we have*

$$\int_r^R |f'(x)|^p (f(x))^{\theta p} dx \leq \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \int_r^R \left(\sqrt{|f(x)f''(x)|}\right)^p (f(x))^{\theta p} dx + \Theta(r,R), \tag{3.3}$$

where $\Theta(r,R) = \frac{1}{\theta p+1}\left(\Phi_p(f'(R))(f(R))^{\theta p+1} - \Phi_p(f'(r))(f(r))^{\theta p+1}\right)$.

We return to the proof of Theorem 3.1, where assumptions: $f \geq \eta$ and $f \in W^{2,1}_{loc}((a,b))$ are relaxed.
Let r, R be such that $a < r < R < b$. Let us consider the following decomposition

$$I_f(r,R) := (r,R) \cap I_f = \{x \in (r,R) : f(x) > 0\} = \bigcup_k I_k, \tag{3.4}$$

where each I_k is of the form

(A) (α_k, β_k) when $f(\alpha_k) = f(\beta_k) = 0$,

(B) (r, β_k) when $f(r) \neq 0$, $f(\beta_k) = 0$,

(C) (α_k, R) when $f(\alpha_k) = 0$, $f(R) \neq 0$,

(D) (r, R) when $f(r) \neq 0$ and $f(R) \neq 0$.

The proof will now proceed in two steps.

Step 1. We prove a variant of inequality (3.2) with (r, R) substituted by I_k.

We consider two (sufficiently small) sequences of positive numbers $\{\varepsilon_{k,l}\}_{l\in\mathbf{N}}$, $\{\delta_{k,l}\}_{l\in\mathbf{N}}$, converging to zero and apply Lemma 3.1 for function f on the interval $[\alpha_k+\varepsilon_{k,l}, \beta_k-\delta_{k,l}]$. Hence

$$\int_{\alpha_k+\varepsilon_{k,l}}^{\beta_k-\delta_{k,l}} |f'(x)|^p (f(x))^{\theta p} dx \leq$$

$$\left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \int_{\alpha_k+\varepsilon_{k,l}}^{\beta_k-\delta_{k,l}} \left(\sqrt{|f(x)f''(x)|}\right)^p |f(x)|^{\theta p} dx + \Theta(\alpha_k+\varepsilon_{k,l}, \beta_k-\delta_{k,l}),$$

where

$$\Theta(\alpha_k+\varepsilon_{k,l}, \beta_k-\delta_{k,l}) = \mathcal{A}f(\beta_k-\delta_{k,l}) - \mathcal{A}f(\alpha_k+\varepsilon_{k,l}).$$

We will let $l \to \infty$. Obviously we have

$$\lim_{l\to\infty} \int_{\alpha_k+\varepsilon_{k,l}}^{\beta_k-\delta_{k,l}} |f'(x)|^p (f(x))^{\theta p} dx = \int_{\alpha_k}^{\beta_k} |f'(x)|^p (f(x))^{\theta p} dx$$

and

$$\lim_{l\to\infty} \int_{\alpha_k+\varepsilon_{k,l}}^{\beta_k-\delta_{k,l}} \left(\sqrt{|f(x)f''(x)|}\right)^p |f(x)|^{\theta p} dx = \int_{\alpha_k}^{\beta_k} \left(\sqrt{|f(x)f''(x)|}\right)^p |f(x)|^{\theta p} dx.$$

Now we verify the convergence of $\Theta(\alpha_k+\varepsilon_{k,l}, \beta_k-\delta_{k,l})$, dealing with each case (A)-(D) separately.

Case (a). Assume first that $\theta > -\frac{1}{p}$.
As $\mathcal{A}f(x)$ has the same sign as $f'(x)$ and f has local minima at α_k and β_k, we can find numbers $\varepsilon_{k,l} \to 0$, $\delta_{k,l} \to 0$ as $l \to \infty$ such that $f'(\alpha_k + \varepsilon_{k,l}) \geq 0$, $f'(\beta_k - \delta_{k,l}) \leq 0$ for every k, l. Therefore obviously

$$\Theta(\alpha_k + \varepsilon_{k,l}, \beta_k - \delta_{k,l}) \leq 0.$$

In case $\theta < -\frac{1}{p}$ we use assumption b) to get:

$$\lim_{l\to 0} \Theta(\alpha_k + \varepsilon_{k,l}, \beta_k - \delta_{k,l}) = 0.$$

Thus in both cases we have

$$\int_{\alpha_k}^{\beta_k} |f'(x)|^p (f(x))^{\theta p} dx \leq \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \int_{\alpha_k}^{\beta_k} \left(\sqrt{|f(x)f''(x)|}\right)^p |f(x)|^{\theta p} dx,$$

and so the estimate (3.2) holds.
Case (b).
When $I_k = (r, \beta_k)$, we get for suitably chosen positive numbers $\delta_{k,l}$ converging to zero

$$\mathcal{A}f(\beta_k - \delta_{k,l}) \leq 0 \quad \text{when} \quad \theta > -\frac{1}{p},$$
$$\mathcal{A}f(\beta_k - \delta_{k,l}) \stackrel{l\to\infty}{\to} 0 \quad \text{when} \quad \theta < -\frac{1}{p}.$$

By similar arguments as in Case (a) but applied to intervals $(r, \beta_k - \delta_{k,l})$ we obtain

$$\int_{r}^{\beta_k} |f'(x)|^p (f(x))^{\theta p} dx \leq \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \int_{r}^{\beta_k} \left(\sqrt{|f(x)f''(x)|}\right)^p |f(x)|^{\theta p} dx - \mathcal{A}f(r),$$

which is (3.2) in this case.
Case (c). For $I_k = (\alpha_k, R)$, (3.2) is obtained by similar arguments and reads as:

$$\int_{\alpha_k}^{R} |f'(x)|^p (f(x))^{\theta p} dx \leq \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \int_{r}^{\beta_k} \left(\sqrt{|f(x)f''(x)|}\right)^p |f(x)|^{\theta p} dx + \mathcal{A}f(R).$$

Case (d).
We apply Lemma 3.1 directly.
Step 2. Summing up these inequalities with respect to k or dealing with Case (d), we get

$$\int_{(r,R)\cap\{x:f(x)>0\}} |f'(x)|^p (f(x))^{\theta p} dx \leq$$
$$\left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \int_{(r,R)\cap\{x:f(x)>0\}} \left(\sqrt{|f(x)f''(x)|}\right)^p |f(x)|^{\theta p} dx + \mathcal{A}f(R) - \mathcal{A}f(r).$$

which ends the proof of the statement. □

Our next statement is an obvious consequence of Theorem 3.1.

Corollary 3.1. *Let the assumptions of Theorem 3.1 be satisfied (in particular $\theta \neq -\frac{1}{p}$) and*

$$\liminf_{R\nearrow b} \mathcal{A}f(R) - \limsup_{r\searrow a} \mathcal{A}f(r) \leq 0.$$

Then we have

$$\int_{\{x\in(a,b):f(x)>0\}} |f'(x)|^p (f(x))^{\theta p} dx \leq \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \int_{\{x\in(a,b):f(x)>0\}} \left(\sqrt{|f(x)f''(x)|}\right)^p (f(x))^{\theta p} dx. \tag{3.5}$$

Remark 3.3. We do not know if the inequality (3.5) holds for $\theta = -\frac{1}{p}$. However, we have weaker inequality for $f \in W^{2,1}_{loc}((a,b))$ (see [11, Proposition 6.3]):

$$\int_{\{x\in(a,b):f(x)\neq 0\}} \frac{|f'|^p}{|f|} dx \leq \left(\sqrt{p-1}\right)^p \int_{\{x\in(a,b):f(x)\neq 0\}} \frac{|ff''\ln(|f|)|^{\frac{p}{2}}}{|f|} dx. \tag{3.6}$$

Remark 3.4. Suppose that we would like to relax the nonnegativity assumption on f in Theorem 3.1. If $\theta \leq -\frac{1}{p}$, $f \in W^{2,1}_{loc}((a,b))$ and f has a single zero in (a,b) then the inequality (3.5) cannot hold with the left-hand side finite when f' is continuous on (a,b) and locally absolutely continuous on (a,b), no matter what boundary conditions are required. Indeed, in such a case we have

$$f(x) = (x-x_0)\omega(x), \text{ where } \omega(x) = \frac{1}{x-x_0}\int_{x_0}^{x} f'(\tau)d\tau \xrightarrow{x\to x_0} f'(x_0) \neq 0,$$

in a neighborhood $I = (x_0-\varepsilon, x_0+\varepsilon) \subseteq (a,b)$ of x_0, where x_0 is such that $f(x_0) = 0, f'(x_0) \neq 0$. We deduce that $\omega \sim Const \neq 0$ on I and so $f \sim (x-x_0)$, $f' \sim 1$ on I. It follows that there exists constant $C > 0$ such that:

$$\int_{\{x\in(a,b):f(x)\neq 0\}} |f'(x)|^p |f(x)|^{\theta p} dx \geq C \int_{(x_0-\varepsilon, x_0+\varepsilon)} |x-x_0|^{\theta p} dx$$

and the right-hand side is infinite if $\theta p \leq -1$. In case when f is nonnegative such situation cannot happen therefore f cannot have single zeros.

4 First order Poincaré inequalities

In this section we obtain certain nonlinear variants of the Poincaré inequality. As a consequence we will find optimal set X such that the mapping $X \ni f \mapsto T(f)$ belongs to the Sobolev space $W^{1,p}((a,b))$, for T given by (2.1).
In the following lemma we compute the distributional derivative of the composition $f \mapsto T(f)$, where the function f belongs to a suitable nonlinear Beppo–Levi set. We have the following result.

Lemma 4.1. *Let $-\infty \le a < b \le +\infty$, $\theta > -1$, T and τ be defined by (2.1). Moreover, let $f \in C((a,b))$ be nonnegative and $f \in L^{1,1,\theta}_{loc}(I_f)$, where $I_f = \{x \in (a,b) : f(x) > 0\}$. Then $T(f) \in W^{1,1}_{loc}(a,b)$ and*

$$(T(f))' = \tau(f) f' \cdot \chi_{\{x: f(x)>0\}} \tag{4.1}$$

in the distributional sense.
If f is strictly positive then the condition $\theta > -1$ above can be relaxed to $\theta \neq -1$.

For the reader's convenience we submit the proof.

Proof. We start with the proof under the assumption $\theta > -1$.
Obviously $T(f) \in L^1_{loc}(a,b)$ because $\theta + 1 > 0$ and f is continuous on (a,b). We compute its weak derivative from the very definition. Let $\phi \in C_0^\infty((a,b))$, then

$$< (T(f))', \phi > = -\int_a^b T(f)\phi' dx \;\; = -\int_{(a,b)\cap\{x: f(x)>0\}} T(f)\phi' dx = -\sum_k \int_{I_k} T(f)\phi' dx,$$

where I_k are disjoint, open intervals such that $\bigcup_k I_k = (a,b) \cap \{x : f(x) > 0\}$. Every I_k is of the form $I_k = (\alpha, \beta)$ for some $a \le \alpha < \beta \le b$, let $I_{k_\varepsilon} = [\alpha+\varepsilon, \beta-\varepsilon]$. Then we have

$$\int_{I_k} T(f)\phi' dx = \lim_{\varepsilon\to 0} \int_{I_{k_\varepsilon}} T(f)\phi' dx.$$

On the interval $[\alpha+\varepsilon, \beta-\varepsilon]$ function f is strictly positive and so $T(f)$ belongs to the space $W^{1,1}(I_{k_\varepsilon})$ (Lemma 2.1). Therefore

$$\lim_{\varepsilon\to 0} \int_{I_{k_\varepsilon}} T(f)\phi' dx = \lim_{\varepsilon\to 0} \left(T(f)\phi \Big|_{\alpha+\varepsilon}^{\beta-\varepsilon} - \int_{I_{k_\varepsilon}} (\tau(f)f')\,\phi dx \right) \;\; = -\int_{I_k} (\tau(f)f')\,\phi dx. \tag{4.2}$$

The last integral is finite because ϕ is compactly supported, $T(f)$ vanishes on those endpoints of I_k which are inside (a,b) and $\tau(f)f'\chi_{\{f>0\}}$ is locally integrable on (a,b). Hence

$$< (T(f))', \phi >= \int_{(a,b)\cap\{x: f(x)>0\}} (\tau(f)f')\,\phi dx. \tag{4.3}$$

Thus $T(f) \in W^{1,1}_{loc}((a,b))$ and (4.1) follows from (4.3).

When f is strictly positive and $\theta \neq -1$, we still have $T(f) \in L^1_{loc}((a,b))$ and the remaining part of the proof follows by the same arguments dealing with one interval I_k only.

$\square$

Remark 4.1. When we dealt with nonnegative continuous function f possibly having zeroes in (a,b), assumption $\theta > -1$ was required for the continuity of function $T(f)$ at x such that $f(x) = 0$. The last inequality in line (4.2) holds due to that assumption. Indeed, if for example $\beta \neq b$ then ϕ might not vanish at β but $T(f(\beta)) = 0$. When $\theta < -1$ it is not true and $T(f)$ is not continuous. Therefore, in such a case, the conclusion $T(f) \in W^{1,1}_{loc}((a,b))$ cannot hold.

The following statement is a variant of the nonlinear Poincaré inequality.

Theorem 4.1. *Let $-\infty < a < b < +\infty$, $p \geq 1$, $\theta \in \mathbb{R}$, $f \in C((a,b))$ be nonnegative and one of the assumptions* **(A)** *or* **(B)** *holds where*

(A) $\theta > -1$ *and*

- a_1) $f \in L^{1,p,\theta}(I_f)$ *where* $I_f = \{x \in (a,b) : f(x) > 0\}$,
- a_2) f *is continuous on* $[a,b]$ *and equal zero at at least one of the endpoints* $z \in \{a,b\}$,

(B) $\theta < -1$, f *is strictly positive and*

- b_1) $f \in L^{1,p,\theta}((a,b))$,
- b_2) *both limits* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to b} f(x)$ *exist and belong to* $(0,\infty]$, *moreover* $\lim_{x\to z} f(x) = \infty$ *for at least one of the endpoints* $z \in \{a,b\}$.

Then $f^{1+\theta} \in W^{1,p}((a,b))$, $(f^{1+\theta})' = (1+\theta) f^\theta f' \cdot \chi_{\{x: f(x)>0\}}$ *and we have*

$$\int_{(a,b)\cap\{f(x)>0\}} (f(x))^p (f(x))^{\theta p} dx \leq C^p_{p,\theta}(a,b) \int_{(a,b)\cap\{f(x)>0\}} |f'(x)|^p (f(x))^{\theta p} dx \qquad (4.4)$$

where $C_{p,\theta}(a,b) = C_p(a,b)|1+\theta|$ *and* $C_p(a,b)$ *is the constant from Proposition 2.1, part i).*

Proof. Assume first **(A)**. According to Lemma 4.1 we have $T(f) \in W^{1,1}_{loc}((a,b))$ and $(T(f))' = \tau(f) f' \chi_{f>0} \in L^p((a,b))$. Therefore the statement follows from Proposition 2.1, part i) applied to $T(f)$.
In case **(B)** due to condition b_2) function $T(f)$ obeys assumptions of Proposition 2.1, part i).

$\square$

Remark 4.2. As follows from the above proof and Remark 2.1, assumption: $f^{\theta+1} \in C([a,b])$ in **(A)** and **(B)** can be relaxed to $f \in C((a,b))$ and conditions $a_2)$ and $b_2)$ substituted by
$a_2^{'})$ f is continuous and equal zero at at least one of the endpoints $z \in \{a,b\}$,
$b_2^{'})$ $\lim_{x\to z} f(x) = \infty$ for at least one of the endpoints $z \in \{a,b\}$.

Remark 4.3. Assume that $\theta > -1$ and consider the mapping

$$f \mapsto T(f) = \frac{1}{1+\theta} \cdot f^{1+\theta}$$

restricted to nonnegative continuous functions on $[a,b]$ the set of which we will denote by $C_{\geq}([a,b])$. Lemma 4.1 and the verification of the L^p integrability show that $T : L^{1,p,\theta}((I_f)) \cap C_{\geq}([a,b]) \mapsto W^{1,p}((a,b)) \cap C_{\geq}([a,b])$ is well defined. Moreover, it is one-to-one. It is also surjective as for any $w \in W^{1,p}((a,b)) \cap C_{\geq}([a,b])$ mapping $f := w^{\frac{1}{1+\theta}}$ belongs to $C_{\geq}([a,b]) \cap W^{1,1}_{loc}(I_f)$ according to Lemma 2.1. By Lemma 4.1 f' can be computed almost everywhere and $(f^{1+\theta})' = (1+\theta) f^{\theta} f' = w' \in L^p((a,b))$. This shows that set $X = L^{1,p,\theta}(I_f) \cap C_{\geq}([a,b])$ is the optimal set for which property

$$X \ni f \mapsto f^{1+\theta} \in W^{1,p}((a,b)) \cap C_{\geq}([a,b]), \tag{4.5}$$

holds. For some selected works dealing with compositions of Sobolev functions see e.g. [3, 6] and their references.

Remark 4.4. $T(f)$ must vanish at at least one of the endpoints of the interval as for example for constant positive functions inequality (4.4) does not hold.

5 Second order Poincaré inequalities

We are now to derive second order nonlinear Poincaré inequalities.

Theorem 5.1. *Suppose that $-\infty < a < b < \infty$, $p \geq 2$, $\theta \in \mathbb{R}$, $\theta \notin \{-\frac{1}{p}, -1\}$, $f \in C((a,b))$ is nonnegative,*

$$\mathcal{A}f(x) := \frac{1}{\theta p + 1} \Phi_p(f'(x))(f(x))^{\theta p+1} \chi_{\{f(x)>0\}},$$

and one of conditions **(C)** *or* **(D)** *holds where*

(C) *$\theta > -1$ and*

> $c_1)$ *$f \in W^{2,1}_{loc}(I_f)$ where $I_f = \{x \in (a,b) : f(x) > 0\}$,*
>
> $c_2)$ *f is continuous and equal zero at at least one of the endpoints $z \in \{a,b\}$,*

c_3) $\liminf_{R\nearrow b}\mathcal{A}f(R) - \limsup_{r\searrow a}\mathcal{A}f(r) \le 0$,

c_4) *in case* $\theta < -\frac{1}{p}$ *function* f *is either strictly positive on* (a,b) *or function* $\mathcal{A}f(\lambda)$ *is continuous on* (a,b).

(D) $\theta < -1$, f *is strictly positive on* (a,b) *and*

d_1) $f \in W^{2,1}_{loc}((a,b))$,

d_2) $\lim_{x\to z} f(x) = \infty$ *for at least one of the endpoints* $z \in \{a,b\}$,

d_3) *condition* c_3) *holds, equivalently*

$$\limsup_{R\nearrow b}\Phi_p(f'(R))(f(R))^{\theta p+1} - \liminf_{r\searrow a}\Phi_p(f'(r))(f(r))^{\theta p+1} \ge 0.$$

Then we have

$$\int_{(a,b)\cap\{x:f(x)>0\}} (f(x))^p(f(x))^{\theta p}dx \le \widetilde{A}\int_{(a,b)\cap\{x:f(x)>0\}} |f''(x)|^p(f(x))^{\theta p}dx \qquad (5.1)$$

$$\int_{(a,b)\cap\{x:f(x)>0\}} |f'(x)|^p(f(x))^{\theta p}dx \le \widetilde{B}\int_{(a,b)\cap\{x:f(x)>0\}} |f''(x)|^p(f(x))^{\theta p}dx \qquad (5.2)$$

where

$$\widetilde{A} = \left(C_p^2(a,b)(1+\theta)^2\cdot\frac{p-1}{|1+\theta p|}\right)^p, \quad \widetilde{B} = \left(C_p(a,b)|1+\theta|\cdot\frac{p-1}{|1+\theta p|}\right)^p$$

and $C_p(a,b)$ *is the constant from Proposition 2.1, part i).*

Proof. Let r and R be such that $a < r < R < b$ and for simplicity we denote

$$\begin{aligned}
A(r,R) &:= \int_{(r,R)\cap\{f>0\}} (f(x))^p(f(x))^{\theta p}dx, \qquad (5.3)\\
B(r,R) &:= \int_{(r,R)\cap\{f>0\}} |f'(x)|^p(f(x))^{\theta p}dx,\\
C(r,R) &:= \int_{(r,R)\cap\{f>0\}} |f''(x)|^p(f(x))^{\theta p}dx,\\
A &:= A(a,b),\ B := B(a,b),\ C := C(a,b)\\
\tilde{\Theta}(r,R) &:= \frac{1}{\theta p+1}\left(\Phi_p(f'(R))(f(R))^{\theta p+1}\chi_{\{f(R)>0\}} - \Phi_p(f'(r))(f(r))^{\theta p+1}\chi_{\{f(r)>0\}}\right)\\
&= \mathcal{A}f(R) - \mathcal{A}f(r),\\
\tilde{\Theta} &:= \liminf_{R\nearrow b, r\searrow a}\tilde{\Theta}(r,R).
\end{aligned}$$

We may assume that $C < \infty$ and $0 < B \le \infty$ as otherwise inequalities follow trivially (in case $B = 0$ we have $f \equiv 0$).

We divide the proof into four steps.

STEP 1. We first show that

$$B(r,R) \le \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} (A(r,R))^{\frac{1}{2}} \cdot (C(r,R))^{\frac{1}{2}} + \tilde{\Theta}(r,R), \tag{5.4}$$

in particular when only $A(r,R)$ is finite, then $B(r,R)$ is finite as well.

Indeed, by our assumptions $(c_1),\ c_4),\ d_1))$ we may apply Theorem 3.1. This together with a (weighted) Schwarz inequality gives

$$\begin{aligned}
&\int_{(r,R)\cap\{f>0\}} |f'(x)|^p (f(x))^{\theta p} dx \\
&\qquad \le \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \int_{(r,R)\cap\{f>0\}} \left(\sqrt{|f(x)f''(x)|}\right)^p (f(x))^{\theta p} dx + \tilde{\Theta}(r,R) \\
&\qquad \le \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \left(\int_{(r,R)\cap\{f>0\}} |f''(x)|^p (f(x))^{\theta p} dx\right)^{\frac{1}{2}} \\
&\qquad\quad \cdot \left(\int_{(r,R)\cap\{f>0\}} (f(x))^p (f(x))^{\theta p} dx\right)^{\frac{1}{2}} + \tilde{\Theta}(r,R),
\end{aligned}$$

which is exactly (5.4).

STEP 2. We now prove that under assumption $A < \infty$ the inequalities (5.1), (5.2) are satisfied.

By our boundary conditions we can chose suitable sequences $r_n \searrow a$ and $R_n \nearrow b$ such that (with some of the limits being possibly infinite)

$$\begin{aligned}
\lim_{n\to\infty} \tilde{\Theta}(r_n, R_n) = \tilde{\Theta} \le 0, \lim_{n\to\infty} A(r_n, R_n) = A, \\
\lim_{n\to\infty} B(r_n, R_n) = B, \lim_{n\to\infty} C(r_n, R_n) = C.
\end{aligned} \tag{5.5}$$

Therefore, since (5.4) is satisfied for every $a < r_n < R_n < b$, we get after passing to the limit with $n \to \infty$:

$$B \le \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} A^{\frac{1}{2}} C^{\frac{1}{2}}. \tag{5.6}$$

Hence $0 < B < \infty$. Therefore $f \in L^{1,p,\theta}(I_f)$ and we may apply Theorem 4.1 to get:

$$A \le C^p_{p,\theta}(a,b) B. \tag{5.7}$$

This combined with (5.6) gives

$$B \le \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} C^{\frac{p}{2}}_{p,\theta}(a,b) B^{\frac{1}{2}} C^{\frac{1}{2}}, \tag{5.8}$$

which after dividing by $B^{\frac{1}{2}}$ and using (5.7) gives

$$\begin{aligned} B &\leq \left(\frac{p-1}{|1+\theta p|} |1+\theta| C_p(a,b) \right)^p C, \\ A &\leq \left(|1+\theta| C_p(a,b) \right)^p B \leq \left(\frac{p-1}{|1+\theta p|} |1+\theta|^2 C_p^2(a,b) \right)^p C \end{aligned}$$

and completes the proof of Step 2. Assumption $A < \infty$ was needed to deduce that $B < \infty$ and to be able to divide by $B^{\frac{1}{2}}$ in the inequality (5.8).

Step 3. Let us now show that for any $-\infty < a < r < R < b < \infty$, and any $z \in (r,R)$

$$\int_{\{(r,R)\cap\{f>0\}\}} (f(x))^{p(1+\theta)} dx \leq \tag{5.9}$$
$$2^{p-1}(R-r)^p \int_{\{(r,R)\cap\{f>0\}} |f'(x)|^p (f(x))^{\theta p} dx + 2^{p-1}(R-r)(f(z))^{(1+\theta)p}.$$

To see this, let $\widetilde{T}(x) = (f(x))^{(1+\theta)} \chi_{\{f(x)>0\}}$ and assume that the right-hand side above is finite. By Lemma 4.1 we have $\widetilde{T} \in W^{1,1}_{loc}((r,R))$ (as $B(r,R) < \infty$) and so for any $z \in (r,R)$.

$$\begin{aligned} |\widetilde{T}(x)| \leq & |\widetilde{T}(x) - \widetilde{T}(z)| + |\widetilde{T}(z)| \leq \left| \int_z^x \widetilde{T}'(s) ds \right| + \widetilde{T}(z) \\ \leq & (R-r) \left(\left(\frac{1}{R-r} \int_r^R |\widetilde{T}'(s)|\, ds \right)^p \right)^{\frac{1}{p}} + \widetilde{T}(z) \\ \leq & (R-r) \left(\frac{1}{R-r} \int_r^R |\widetilde{T}'(s)|^p ds \right)^{\frac{1}{p}} + \widetilde{T}(z) \\ = & (R-r)^{1-\frac{1}{p}} \left(\int_r^R |\widetilde{T}'(s)|^p ds \right)^{\frac{1}{p}} + \widetilde{T}(z). \end{aligned}$$

For any $p \geq 1$, $\alpha, \beta \geq 0$, we have $(\alpha+\beta)^p \leq 2^{p-1}(\alpha^p + \beta^p)$. Therefore

$$(\widetilde{T}(x))^p \leq 2^{p-1}(R-r)^{p-1} \int_r^R |\widetilde{T}'(s)|^p ds + 2^{p-1}(\widetilde{T}(z))^p. \tag{5.10}$$

According to Lemma 4.1 we have $\widetilde{T}'(x) = (f(x))^\theta \cdot f' \cdot \chi_{\{x:f(x)>0\}}$. Now (5.9) follows after integrating (5.10) over (r,R).

Step 4. We relax on assumption $A < \infty$ from Step 2 and finish the proof. For this, first we show that B in (5.3) must be finite.

According to Step 3 we have

$$A(r,R) \le \widetilde{C_1}(r,R)B(r,R) + \widetilde{C_2}(r,R)(f(z))^{(1+\theta)p},$$

where

$$\widetilde{C_1}(r,R) = 2^{p-1}(R-r)^p, \quad \widetilde{C_2}(r,R) = 2^{p-1}(R-r).$$

Combining this with (5.4) gives

$$\begin{aligned} B(r,R) &\le \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} A^{\frac12}(r,R)\cdot C^{\frac12}(r,R) + \tilde{\Theta}(r,R) \\ &\le \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \left(\widetilde{C_1}(r,R)B(r,R) + \widetilde{C_2}(r,R)(f(z))^{p(\theta+1)}\right)^{\frac12} C^{\frac12}(r,R) + \tilde{\Theta}(r,R) \\ &\le \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \left(\widetilde{C_1}^{\frac12}(r,R)\left(B^{\frac12}(r,R)\right) + \widetilde{C_2}^{\frac12}(r,R)(f(z))^{\frac{p(\theta+1)}{2}}\right) C^{\frac12}(r,R) + \tilde{\Theta}(r,R). \end{aligned}$$

The above inequality is of the form

$$X \le \alpha X^{\frac12} + \beta,$$

where

$$X = B(r,R) \ge 0, \tag{5.11}$$

$$\alpha = \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \widetilde{C_1}^{\frac12}(r,R)C^{\frac12}(r,R), \tag{5.12}$$

$$\beta = \left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \widetilde{C_2}^{\frac12}(r,R)(f(z))^{\frac{p|\theta+1|}{2}} C^{\frac12}(r,R) + \tilde{\Theta}(r,R). \tag{5.13}$$

It implies

$$X^{\frac12} \le \frac{\alpha + \sqrt{\alpha^2+4\beta}}{2}.$$

Consequently

$$X \le \alpha^2 + 2\beta \tag{5.14}$$

and α, β can be globally estimated independently of r, R. Let $r_n \searrow a$, $R_n \nearrow b$ be such that $\lim_{n\to\infty} \tilde{\Theta}(r_n, R_n) = \tilde{\Theta} \le 0$. It is clear that

$$\lim_{n\to\infty} C(r_n, R_n) = C < \infty.$$

Thus and by (5.11) and (5.14)

$$\lim_{n\to\infty} B(r_n, R_n) = B < \infty.$$

Using the estimate

$$A(r_n, R_n) \le 2^{p-1}(R-r)^p B(r_n, R_n) + 2^{p-1}(R_n - r_n)(f(z))^{p(\theta+1)}\chi_{\{f(z)>0\}} \le \widetilde{C},$$

we get

$$\lim_{n\to\infty} A(r_n, R_n) = A.$$

We deduce that $A < \infty$ and so we can apply Step 2 to finish the proof. □

Remark 5.1. To apply Step 1 we required assumptions: **(C)**: c_1), c_4) and **(D)**: d_1), for Step 2 we required **(C)**: c_1), c_2), c_3) and **(D)**: d_1), d_2), d_3), while for Steps 3 and 4 we required **(C)**: c_1) and **(D)**: d_1).

6 Elliptic type *a priori* estimates and nonlinear eigenvalue problems

Our main goal is to show an application of the desired inequalities to obtain new *a priori* estimates to certain second order quasilinear singular ODEs. For this purpose, we consider the following problem:

$$f''(x) = g(x)(f(x))^{-\theta} \text{ for } x \in (a,b),\ f(x) \ge 0,\ \theta \in \mathbb{R}, \tag{6.1}$$

where $-\infty < a < b < \infty$, $g \in L^p((a,b))$, subject to boundary condition $f \in \mathcal{R}$ which will be introduced later.

We provide below an analysis of the ODE (6.1), ending with the precise definition of its solution. Contrary to the classical approach the solution *may not* be an element of the classical Sololev space, but may rather be a composition of Sobolev function.

Analysis of ODE (6.1)**.**
Under the assumption $f \in C((a,b))$ the left-hand side has interpretation in $\mathcal{D}'((a,b))$, but for the right-hand side this is not always the case. For further analysis we consider two situations:
1) $\theta \le 0$. In this case, as $g \in L^p((a,b))$ and f is locally bounded on (a,b), the right hand side is a locally integrable function, so it defines a distribution on (a,b). In particular (6.1) has a good interpretation in the distributional sense. Moreover, in this case we have $f \in W^{2,p}_{loc}((a,b))$ (see [17], Section 1.1.2).
2) $\theta > 0$. In this case the right-hand side is well defined *only* on the open domain $I_f = \{x \in (a,b) : f(x) > 0\}$, as otherwise $(f(x))^{-\theta}$ is not defined. Moreover, the right-hand side in (6.1) is locally integrable on I_f. Therefore (6.1) has a good interpretation in the space of distributions on I_f and I_f is the *optional domain* for the validity of (6.1). Note that when interpreting (6.1) in $\mathcal{D}'(I_f)$, we automatically deduce $f \in W^{2,p}_{loc}(I_f)$.

The above analysis leads to the following definition of solutions to (6.1).

Definition 6.1. Let $-\infty < a < b < \infty$ and $g \in L^p((a,b))$, $\theta \in \mathbb{R}$. We will say that f is a solution to

$$f''(x) = g(x)(f(x))^{-\theta} \text{ for } x \in (a,b),\ \ f(x) \geq 0,$$

if f is nonnegative and the following assumptions hold:
1) $f \in C((a,b))$, in particular f'' is well defined in the sense of distributions;
2) in case $\theta \leq 0$ we assume that $f \in W^{2,1}_{loc}((a,b))$ and (6.1) holds in $\mathcal{D}'((a,b))$, while in case $\theta > 0$ we assume $f \in W^{2,1}_{loc}(I_f)$ where $I_f = \{x \in (a,b) : f(x) > 0\}$ and the equation holds in $\mathcal{D}'(I_f)$.

An example of an ODE of the form (6.1) where f is not necessarily strictly positive and *does not* belong to $W^{2,1}_{loc}((a,b))$ is given below.

Example 6.1. Let $(a,b) = (-1,1)$ and $f(x) = x^\alpha \chi_{x>0}$ where $\alpha \in (0,1)$. Then $I_f = (0,1)$ and $f \in W^{2,1}(I_f)$ but $f \notin W^{2,1}_{loc}((-1,1))$. Nevertheless, we have on $(0,1)$

$$f''(x) = \alpha(\alpha-1)x^{\alpha-2} = \alpha(\alpha-1)x^{\alpha-2+\alpha\theta}\chi_{x>0} \cdot (x^\alpha)^{-\theta}\chi_{x>0} =: g_\theta(x)(f(x))^{-\theta}$$

and $g_\theta \in L^p(-1,1)$ whenever $\theta < -1 + \frac{1}{\alpha}(2 - \frac{1}{p}) = \kappa_p \in (1 - \frac{1}{p}, \infty)$. We observe that f is not positive on the whole set $(-1,1)$ and Definition 6.1 is satisfied. In particular Harnack principle in general does not hold for such solutions.

Our main result reads as follows.

Theorem 6.1. *Suppose that $-\infty < a < b < \infty$, $p \geq 2$, $\theta \in \mathbb{R}$, $\theta \notin \{-\frac{1}{p}, -1\}$, $\int_a^b |g(x)|^p dx < \infty$, $f \in C((a,b))$ is the nonnegative solution of* (6.1) *in the sense of Definition 6.1, and one of the conditions* **(C)** *or* **(D)** *holds where*

(C) *$\theta > -1$ and*

- $c_1)$ *f is continuous and equal zero at at least one of the endpoints $z \in \{a,b\}$,*
- $c_2)$
$$\liminf_{R \nearrow b} \mathcal{A}f(R) - \limsup_{r \searrow a} \mathcal{A}f(r) \leq 0,$$
where $\mathcal{A}f(x) := \frac{1}{\theta p+1}\Phi_p(f'(x))(f(x))^{\theta p+1}\chi_{\{f>0\}}$,
- $c_3)$ *in case $\theta < -\frac{1}{p}$ function f is either strictly positive or function $\mathcal{A}f(x)$ is continuous on (a,b).*

(D) *$\theta < -1$, f is strictly positive on (a,b) and*

$d_1)$ $\lim_{x\to z} f(x) = \infty$ *for at least one of the endpoints* $z \in \{a, b\}$,

$d_2)$ *condition* $c_2)$ *is satisfied, equivalently*

$$\limsup_{R\nearrow b} \Phi_p(f'(R))(f(R))^{\theta p+1} - \liminf_{r\searrow a} \Phi_p(f'(r))(f(r))^{\theta p+1} \geq 0.$$

Then $f \in L^{2,p,\theta}(I_f)$, $f^{1+\theta} \in W^{1,p}((a,b))$ *and we have:*

i)

$$\int_a^b |f(x)|^p (f(x))^{\theta p} dx \leq A_g, \tag{6.2}$$

$$\int_{(a,b)\cap\{x:f(x)>0\}} |f'(x)|^p (f(x))^{\theta p} dx \leq B_g, \tag{6.3}$$

$$\int_{(a,b)\cap\{x:f(x)>0\}} |f''(x)|^p (f(x))^{\theta p} dx = C_g, \tag{6.4}$$

where

$$A_g = \left\{ C_p^2(a,b)(1+\theta)^2 \frac{p-1}{|1+\theta p|} \right\}^p \int_{(a,b)\cap\{x:f(x)>0\}} |g(x)|^p \, dx,$$

$$B_g = \left\{ C_p(a,b)|1+\theta| \frac{p-1}{|1+\theta p|} \right\}^p \int_{(a,b)\cap\{x:f(x)>0\}} |g(x)|^p \, dx,$$

$$C_g = \int_{(a,b)\cap\{x:f(x)>0\}} |g(x)|^p dx.$$

ii)

$$\sup \left\{ \frac{|f^{1+\theta}(x) - f^{1+\theta}(y)|}{|x-y|^{1-\frac{1}{p}}} : x, y \in (a,b) \right\} \leq D_p \left(\int_a^b |g(x)|^p dx \right)^{\frac{1}{p}},$$

where $D_p = C_p(a,b)|1+\theta|^{1+\frac{1}{p}} \frac{p-1}{|1+\theta p|}$.

iii) *for* $\theta > -1$ *and* $\lim_{x\to a} f(x) =: f(a)$, *we have*

$$f(x) \leq \left\{ f^{1+\theta}(a) + |x-a|^{1-\frac{1}{p}} D_p \left(\int_a^b |g(x)|^p dx \right)^{\frac{1}{p}} \right\}^{\frac{1}{1+\theta}}.$$

Proof.

i) Inequality (6.4) is obvious since

$$C_g = \int_{(a,b)\cap\{x: f(x)>0\}} |g(x)|^p dx = \int_{(a,b)\cap\{x: f(x)>0\}} |f''(x)|^p (f(x))^{\theta p} dx,$$

while (6.2) and (6.3) follow from Theorem 5.1. The fact that $f^{\theta+1} \in W^{1,p}((a,b))$ follows from Theorem 4.1.

ii) As we have $F = f^{1+\theta} \in W^{1,p}(a,b)$, it suffices to apply the Morrey–Sobolev inequality ([17, Theorem 1.4.5, part (f)]):

$$\frac{|F(x)-F(y)|}{|x-y|^{1-\frac{1}{p}}} \leq \frac{1}{|x-y|^{1-\frac{1}{p}}} \int_x^y |F'(x)|dx \leq \frac{|x-y|^{1-\frac{1}{p}}}{|x-y|^{1-\frac{1}{p}}} \left(\int_x^y |F'(x)|^p dx\right)^{\frac{1}{p}},$$

where $x, y \in (a,b), x < y$.

iii) We leave this part to the reader as an easy exercise.

□

Remark 6.1. The following example shows that one can find function f obeying assumptions in Theorem 6.1 and $f^{1+\theta} \in W^{1,p}((a,b))$ but $f^{1+\theta} \notin W^{2,p}((a,b))$. Indeed, consider the same function as in Example 6.1, namely let $(a,b) = (-1,1)$, $f_\alpha(x) = x^\alpha \chi_{x>0}$, where parameter $\alpha > 0$ will be estabilished later. By the same verification we have $f_\alpha''(x) = g_{\alpha,\theta}(x)(f_\alpha)^\theta$ where $g_{\alpha,\theta}(x) = \alpha(\alpha-1)x^{\alpha-2-\alpha\theta}\chi_{x>0}$. We note that:

$$g_{\alpha,\theta} \in L^p((-1,1)) \Longleftrightarrow \theta < -1 + \frac{1}{\alpha}(2 - \frac{1}{p}) = \kappa_p(\alpha);$$
$$(f_\alpha)^{1+\theta} \in W^{1,p}((-1,1)) \Longleftrightarrow \theta > -1 + \frac{1}{\alpha}(1 - \frac{1}{p}) = \beta_p(\alpha);$$
$$(f_\alpha)^{1+\theta} \in W^{2,p}((-1,1)) \Longleftrightarrow \theta > -1 + \frac{1}{\alpha}(2 - \frac{1}{p}) = \kappa_p(\alpha);$$

and $\beta_p(\alpha) < \kappa_p(\alpha)$. Choosing $\alpha \in \left(\frac{p-1}{p(\theta+1)}, \frac{2-p}{p(\theta+1)}\right)$ we have $f_\alpha \in W^{2,1}_{loc}((-1,1))$ and $f_\alpha^{1+\theta} \in W^{1,p}((-1,1))$ but $f_\alpha^{1+\theta}$ cannot belong to $W^{2,p}((-1,1))$. In that sense regularity result $f_\alpha^{1+\theta} \in W^{1,p}((-1,1))$ cannot be improved to $f_\alpha^{1+\theta} \in W^{2,p}((-1,1))$.
This also shows that in general the statement:
$f_\alpha \in L^{2,p,\theta}(I_f) \cap C_{\geq 0}([-1,1]) \Longleftrightarrow f_\alpha^{1+\theta} \in W^{2,p}((-1,1)) \cap C_{\geq 0}([-1,1])$
is false, while the statement:
$f_\alpha \in L^{1,p,\theta}(I_f) \cap C_{\geq 0}([-1,1]) \Longleftrightarrow f_\alpha^{1+\theta} \in W^{1,p}((-1,1)) \cap C_{\geq 0}([-1,1])$
is true (see Remark 4.3).

7 Additional remarks

The following remarks are in order.

Remark 7.1. When $\theta > -1$ conditions c_1) and c_2) hold if we assume that f is nonnegative and satisfies Dirichlet boundary condition: $f(a) = f(b) = 0$. When $\theta < -1$ $d_1), d_2$) hold provided that $\lim_{x\to a} f(x) = \lim_{x\to b} f(x) = \infty$.

Remark 7.2. Theorem 6.1 can be compared with the following statement obtained in [11, Proposition 7.2], dealing with slightly different boundary conditions and essentially stronger assumptions: f is strictly positive, $f \in W^{2,1}_{loc}((a,b))$ (in this case $a = 0$).

Proposition 7.1. *Suppose that* $1 \le q < \infty$*,* $\alpha \neq -1 + \frac{1}{q}$*,* $\kappa = -\mathrm{sign}(\alpha + 1 - \frac{1}{q})$*,* $0 < b \le \infty$*,* $g \in L^q((0,b))$ *and let* $f \in W^{2,1}_{loc}((0,b))$ *be a positive solution of the following ODE:*

$$\begin{cases} f''(x) = g(x)(f(x))^{\alpha} \text{ a.e. on } (0,b) \\ \liminf\limits_{R\nearrow b} \kappa |f'(R)|^{2q-2} f'(R)(f(R))^{-q(\alpha+1)+1} - \limsup\limits_{r\searrow 0} \kappa |f'(r)|^{2q-2} f'(r)(f(r))^{-q(\alpha+1)+1} \le 0. \end{cases}$$

Then we have

i) $|f'|^{2q}|f|^{-q(\alpha+1)} \in L^1(0,b)$ *and*

$$\int_0^b |f'(x)|^{2q}|f(x)|^{-q(\alpha+1)} dx \;\le\; \left(\frac{(2q-1)}{|q-1+\alpha q|}\right)^q \int_0^b |g(x)|^q dx,$$

ii)

$$\sup\left\{ \frac{|(f(x))^{\frac{1-\alpha}{2}} - (f(y))^{\frac{1-\alpha}{2}}|}{|x-y|^{1-\frac{1}{2q}}} : x,y \in (0,b) \right\} \le A_q \left(\int_0^b |g(x)|^q dx \right)^{\frac{1}{2q}},$$

$$A_q = \sqrt{2q-1}\,|q-1+\alpha q|^{-\frac{1}{2}} \frac{|1-\alpha|}{2},$$

iii) *If* $\alpha < 1$ *then* $\lim_{r\searrow 0} f(r) =: f(0)$ *exists and the following estimate holds:*

$$|f(x)| \;\le\; \left\{ (f(0))^{\frac{1-\alpha}{2}} + A_q |x|^{1-\frac{1}{2q}} \left(\int_0^b |g(x)|^q dx \right)^{\frac{1}{2q}} \right\}^{\frac{2}{1-\alpha}}.$$

Remark 7.3. In the original statement of assertion iii) in [11, Proposition 7.2] there is an estimate:

$$|f(x)| \;\le\; \left\{ (f(0))^{\frac{1-\alpha}{2}} + A_q |x|^{1-\frac{1}{2q}} \left(\int_0^b |g(x)|^q dx \right)^{\frac{1}{q}} \right\}^{\frac{2}{1-\alpha}},$$

involving $\left(\int_0^b |g(x)|^q dx\right)^{\frac{1}{q}}$ instead of $\left(\int_0^b |g(x)|^q dx\right)^{\frac{1}{2q}}$. This is a typo as iii) follows trivially from ii).

Remark 7.4. In the paper by Adamowicz and the first author [1], the authors consider the monotonicity properties of radial solutions to PDE having general form:

$$-a(|x|)\Delta_q(w) + h(|x|, w, \nabla w \cdot \frac{x}{|x|}) = \phi(w),$$

where $\Delta_q(w) = -\mathrm{div}\,(|\nabla w|^{q-2}\nabla w)$ is the q-Laplacian and the equation is defined on a ball in $\mathbb{R}^n$. The particular cases that interest us are of the form:

$$-a(|x|)w'' = w^{-\theta}, \text{ equivalently } w'' = -\frac{1}{a(|x|)}w^{-\theta} =: -g(x)w^{-\theta}, \tag{7.1}$$

The adaptation of results from [1, Section 5], allows to deduce that (under some special assumptions) the nonnegative solutions to (7.1) are often monotonic and as a consequence we can in such cases simplify the assumptions $c_2), d_2)$ in Theorem 6.1.

The following statement is obtained by an adaptation of methods from [1].

Theorem 7.1. *Assume that* $f : [0, R] \to [0, \infty)$ *is the solution to* (1.1) *and additionally:*

1) $g \in W^{1,1}_{loc}((0, R)) \cap C([0, R))$, $g > 0$ *in* $[0, R)$ *and* g *is nondecreasing;*
2) $f \in W^{2,1}_{loc}((0, R)) \cap C^1([0, r))$ *for any* $0 < r < R$ *and* $f'(0) \le 0$*;*
3) $-\frac{1}{p} < \theta < 1$.

Then f *is nonincreasing in a neighborhood of* R*. In particular when* $f'(0) = 0$ *then condition* $c_2)$*:*

$$\liminf_{r\nearrow R} \mathcal{A}f(r) - \limsup_{r\searrow 0} \mathcal{A}f(r) \le 0$$

in Theorem 6.1 is satisfied for f*, where* $\mathcal{A}f(x) := \frac{1}{\theta p+1}\Phi_p(f'(x))(f(x))^{\theta p+1}\chi_{\{f>0\}}$*.*
Moreover, if $f(x_0) = 0$ *for some* $x_0 \in [0, R)$ *then* $f(x) \equiv 0$ *for every* $x \ge x_0$*.*

Proof. Assume at first that $f'(0) = 0$.
We have $\left(\frac{1}{2}(f')^2\right)' = -\frac{1}{1-\theta}\left(f^{1-\theta}\right)' g(x)$ and the function $\Psi(s) = \frac{1}{1-\theta}s^{1-\theta}$ is increasing. We consider the following three cases.

Case 1. f is strictly positive on some interval (x, R) where $0 \le x < R$.
Let $x_0 := \inf\{x : f(x) > 0 \text{ on } (x, R)\}$. Then we have either $x_0 = 0$ or $f(x_0) = 0$ for a $x_0 \in (0, R)$. In both cases x_0 is a stationary point for f. We have two situations: a) there are no other stationary points in (x_0, R) and b) set $\{x \in (x_0, R) :$

x is a stationary point for f} is not empty. In case a) it suffices to check that $\Psi(f(x_0)) - \Psi(f(R)) \geq 0$. We have for any $0 < \varepsilon < R - x_0$:

$$\begin{aligned} \Psi(f(x_0)) - \Psi(f(R-\varepsilon)) &= \int_{R-\varepsilon}^{x_0} f(x)^{-\theta} f'(x)dx = \int_{x_0}^{R-\varepsilon} \frac{1}{g(x)} f''(x)f'(x)dx \\ &= \int_{x_0}^{R-\varepsilon} \frac{1}{g(x)} \left(\frac{1}{2}(f'(x))^2\right)' = \frac{1}{g(x)}\frac{1}{2}(f'(x))^2|_{x_0}^{R-\varepsilon} \\ &- \int_{x_0}^{R-\varepsilon} \left(\frac{1}{g(x)}\right)' \frac{1}{2}(f'(x))^2 dx \geq 0 \end{aligned} \tag{7.2}$$

and we can let ε converge to zero. In case b) we consider any $x_0 < y_1 < y_2 \leq R$ and we will show that $f(y_1) \geq f(y_2)$. If y_1 is a stationary point for f this follows by the same computations as in (7.2) with (x_0, R) substituted by (y_1, y_2). If it is not the case, let $\bar{y}_1 < y_1$ be the nearest to y_1 stationary point in $[x_0, y_1)$. Modification of (7.2) shows that then f is decreasing on $(\bar{y}_1, y_1)$, so it is also decreasing in a neighborhood of y_1. If there are no stationary points in (y_1, y_2), we deduce that $f(y_1) > f(y_2)$. If it is not the case, let $\bar{y}_2 \in (y_1, y_2)$ be the nearest stationary point to y_1, then $f(y_1) > f(\bar{y}_2)$. By modification of (7.2) with (x_0, R) substituted by $(\bar{y}_2, y_2)$ we deduce that then $f(\bar{y}_2) \geq f(y_2)$. In all situations we have $f(y_1) \geq f(y_2)$.

Case 2. f is zero in a neighborhood of R.
In this case the assertion holds trivially.

Case 3. There exists a sequence of zeroes of f converging to R and f is not identically zero in every neighborhood of R.
This situation is impossible. Indeed, if it would hold, in arbitrary small neighborhood of R, we could find $0 < x_1 < x_2 \leq R$ such that $f(x_1) = 0$, $f(x_2) > 0$ and $f > 0$ on (x_1, x_2). The computations in (7.2) with (x_0, R) substituted by (x_1, x_2) show that then $0 = f(x_1) \geq f(x_2) > 0$, a contradiction.

When $f'(0) < 0$ then either f has no stationary point in $(0, R)$ and then the assertion is true or there exists a stationary point $\bar{x} \in (0, R)$. In the last situation it suffices to apply previous arguments with 0 substituted by $\bar{x}$. This completes the proof. □

Remark 7.5. Taliaferro in [23] studied the asymptotic behavior of positive classical solutions to equation:

$$\begin{cases} f''(x) + g(x)f^{-\theta}(x) = 0, \quad \text{where} \quad x \in (0,1), \\ f(0) = f(1) = 0 \end{cases} \tag{7.3}$$

where $\theta > 0$, g is positive and continuous in $(0, 1)$. It is proven there (see Theorems 3 and 5) that (7.3) has positive classical solutions if and only if $\int_0^1 t(1-t)g(t)dt < \infty$. Assuming this condition the author showed that

i) if $\int_0^{1/2} g(t)t^{-\theta}dt < \infty$ then there exists positive constant a such that

$$f(t) \sim at - a^{\theta}(1 + o(1)) \int_0^t (t-s)g(s)s^{-\theta}ds,$$

as $t \to 0$;

ii) if $\int_0^{1/2} g(t)t^{-\theta}dt = \infty$ and $h(t) := \left(\int_t^{1/2} \psi(s)s^{-\theta}ds\right)^{\frac{1}{\theta+1}}$ where $\psi > 0$, $\psi \sim g$ as $t \to 0$, $\psi \in C([0,1/2]) \cap C^1((0,1/2))$ and $\lim_{t\to 0} \frac{th''(t)}{h'(t)} =: R > -2$ then

$$f(t) \sim \left(\frac{\theta+1}{2+R}\right)^{\frac{1}{1+\theta}} th(t),$$

as $t \to 0$.

Conditions on g in i) and ii) involve weighted L^1 spaces $L^1((0,1), t(1-t)dt)$ and $L^1((0,1/2), t^{-\theta}dt)$, while we assume that g belongs the unweighted L^p space on the domain of solutions to the ODE. Therefore Taliaferro's results cannot be directly compared with ours.
On the other hand, when $\theta > -1$ and f solves (7.3), we deduce from our Theorem 6.1, statement iii) and Remark 7.1 that solutions satisfy the estimate:

$$0 \le f(x) \le Cx^{\frac{p-1}{p(1+\theta)}} \left(\int_0^1 |g(x)|^p dx\right)^{\frac{1}{p(1+\theta)}}.$$

In particular when $-1 < \theta < -\frac{1}{p}$ we have $\frac{p-1}{p(1+\theta)} > 1$ and our estimate allows to conclude that solutions converge to zero as $x \to 0$ faster than linearly.
It would be interesting to provide analysis on infinite intervals as well and describe the asymptotic behavior of solutions near infinity.

Remark 7.6. Several nonexistence results which can be adapted to general equations like (1.1) can be found in papers [1, 12]. To the best of our knowledge, the existence results of Emden–Fowler equation (1.1) outside continuity assumptions on function g are missing in the literature.

Acknowledgments. Part of this work was done when A.K. visited Istituto per le Applicazioni del Calcolo "Mauro Picone" Consiglio Nazionale delle Ricerche in Naples, in May 2013. She thanks Claudia Capone and Alberto Fiorenza for discussions and hospitality.

References

[1] Adamowicz, T., Kałamajska, A., *Maximum principles and nonexistence results for radial solutions to equations involving p-Laplacian*, Math. Methods Appl. Sci. 33 (2010), no. 13, 1618–1627.

[2] Agarwal, R. P. and O'Regan, D. and Palamides, P. K., *The generalized Thomas-Fermi singular boundary value problems for neutral atoms*. Math. Methods Appl. Sci. 29, 1 (2006), 49–66.

[3] Brezis, H. and Mironescu, P., *Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces*. J. Evol. Equ. 1, 4 (2001), 387–404.

[4] Buttazzo, G. and Giaquinta, M. and Hildebrandt, S., *One-dimensional variational problems. An introduction*. Oxford Lecture Series in Mathematics and its Applications, 15. The Clarendon Press Oxford University Press, New York, 1998.

[5] Cano-Casanova, S., *Decay rate at infinity of the positive solutions of a generalized class of Thomas-Fermi equations*. Discrete Contin. Dyn. Syst., Dynamical systems, differential equations and applications. 8th AIMS Conference. Suppl. Vol. I, 2011, 240–249.

[6] Dahlberg, B.E.J., *A note on Sobolev spaces, in Harmonic Analysis*, in Harmonic Analysis in Euclidean Spaces, Proc. Symp. in Pure Mathematics Part I, American Math. Soc. 35 (1979), 183–185.

[7] Carleman, T., *Problèmes mathématiques dans la théorie cinétique de gaz*. Almqvist & Wiksell, 1957.

[8] Fermi, E., *Un methodo statistico par la determinazione di alcune properitá dell' atoma*, Rend. Accad. Naz. del Lincei Cl. Sci. Fis. Mat. e Nat. **6** (1927), 602–607.

[9] Gatica, J. A., Oliker, V. and Waltman, P. *Singular nonlinear boundary value problems for second-order ordinary differential equations*, J. Differential Equations 79, 1989, 62–78.

[10] Janus, J. and Myjak, J., *A generalized Emden-Fowler equation with a negative exponent*. Nonlinear Anal. 23, 8 (1994), 953–970.

[11] Kałamajska, A. and Peszek, J., *On some nonlinear extensions of the Gagliardo-Nirenberg inequality with applications to nonlinear eigenvalue problems*. Asymptotic Analysis, Volume 77, Number 3-4 (2012), 169–196.

[12] Kałamajska, A., Pietruska-Pałuba, K., Skrzypczak, I., *Nonexistence results for differential inequalities involving A-Laplacian,* Advances in Differential Equations, Volume 17, Numbers 3-4 (2012), 307–336.

[13] Kawohl, B., Lindqvist, P., *Positive eigenfunctions for the p-Laplace operator revisited.* Analysis (Munich) 26 (2006) 539–544.

[14] Luning, C. D. and Perry, W. L., *An iterative method for solution of a boundary value problem in non-newtonian fluid flow.* J. of Non-Newtonian Fluid Mech. 15, 4 (1984), 145–154.

[15] Luning, C. D. and Perry, W. L., *Iterative solution of negative exponent Emden-Fowler problems.* Internat. J. Math. Math. Sci. 13, 1 (1990), 159–164.

[16] Luning, C. D. and Perry, W. L., *Positive solutions of negative exponent generalized Emden-Fowler boundary value problems.* SIAM J. Math. Anal. 12, 6 (1981), 874–879.

[17] Maz'ja, V. G., *Sobolev Spaces.* Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.

[18] Nachman, A. and Callegari, A., *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids.* SIAM J. Appl. Math. 38, 2 (1980), 275-281.

[19] Naito, M., *Asymptotic behavior of solutions of second order differential equations with integrable coefficients.* Trans. Amer. Math. Soc. 282, 2 (1984), 577–588.

[20] Ntouyas, S. K. and Palamides, P. K., *The existence of positive solutions of nonlinear singular second-order boundary value problems.* Math. Comput. Modelling 34, 5-6 (2001), 641–656.

[21] O'Regan, D., *Theory of singular boundary value problems.* World Scientific Publishing Co. Inc., River Edge, NJ, 1994.

[22] Perry, W. L., *A monotone iterative technique for solution of pth order ($p < 0$) reaction-diffusion problems in permeable catalysis.* Journal of Computational Chemistry 5, 4 (1984), 353–357.

[23] Taliaferro, S. D., *A nonlinear singular boundary value problem.* Nonlinear Anal. 3, 6 (1979), 897–904.

[24] Thomas, L.H., *The calculation of atomic fields,* Proc. Camb. Phil. Soc. **23** (1927), 1473–1484.

[25] Tsukamoto, I., *Asymptotic behavior of solutions of $x'' = e^{\alpha\lambda t}x^{1+\alpha}$ where $-1 < \alpha < 0$.* Osaka J. Math. 40, 3 (2003), acm 595–620.

[26] Whitman, G. B., *Linear and nonlinear waves.* Wiley–Interscience, New York, 1974.

[27] Wong, J. S. W., *On the generalized Emden-Fowler equation.*, SIAM Rev. 17, 1975, 339–360.

[28] Wong, J. S. W., *Oscillation criteria for second order nonlinear differential equations with integrable coefficients*, Proc. Amer. Math. Soc. 115, 1992, 389–395.

[29] Zhang, Z., *Asymptotic behavior of the unique solution to a singular elliptic problem with nonlinear convection term and singular weight.* Adv. Nonlinear Stud. 8, 2 (2008), 391–400.